Derivative Pricing with Credit Risk

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ABSTRACT

This article presents a new model for valuing financial contracts subject to credit risk and collateralization. We study credit default swap (CDS) contract subject to counterparty risk. There are three credit risk factors in CDS. They are credit risks from the buyer, seller and reference entity. We show that default dependency has a significant impact on the value of CDS. We also show that a fully collateralized CDS is not equivalent to a risk-free one. In other words, full collateralization cannot eliminate counterparty risk completely in the CDS market.

Key Words: asset pricing; credit risk modeling; collateralization, CDS.

1 Introduction

There are two primary types of models that attempt to describe default processes in the literature: structural models and reduced-form (or intensity) models. Many practitioners in the credit trading arena have tended to gravitate toward the reduced-from models given their mathematical tractability.

Central to the reduced-form models is the assumption that multiple defaults are independent conditional on the state of the economy. In reality, however, the default of one party might affect the default probabilities of other parties.

The main drawback of the conditionally independent assumption or the reduced-form models is that the range of default correlations that can be achieved is typically too low when compared with empirical default correlations (see Das et al. (2007)). The responses to correct this weakness can be generally classified into two categories: endogenous default relationship approaches and exogenous default relationship approaches.

The endogenous approaches include the contagion (or infectious) models and frailty models. Contagion and frailty models fill an important gap but at the cost of analytic tractability. They can be especially difficult to implement for large portfolios.

Given a default model, one can value a risky derivative contract and compute credit value adjustment (CVA) that is a relatively new area of financial derivative modeling and trading. CVA is the expected loss arising from the default of a counterparty.

Collateralization as one of the primary credit risk mitigation techniques becomes increasingly important and widespread in derivatives transactions. According the ISDA (2013), 73.7% of all OTC derivatives trades (cleared ad non-cleared) are subject to collateral agreements. For large firms, the figure is 80.7%. On an asset class basis, 83.0% of all CDS transactions and 79.2% of all fixed income transactions are collateralized. For large firms, the figures are 96.3% and 89.4%, respectively.

This paper presents a new framework for valuing defaultable financial contracts with or without collateral arrangements. The framework characterizes default dependencies exogenously, and models collateral processes directly based on the fundamental principals of collateral agreements. For brevity we focus on CDS contracts, but many of the points we make are equally applicable to other derivatives. CDS has trilateral credit risk, where three parties – buyer, seller and reference entity – are defaultable.

In general, a CDS contract is used to transfer the credit risk of a reference entity from one party to another. The risk circularity that transfers one type of risk (reference credit risk) into another (counterparty credit risk) within the CDS market is a concern for financial stability. Some people claim that the CDS market has increased financial contagion or even propose an outright ban on these instruments.

The standard CDS pricing model in the market assumes that there is no counterparty risk. Although this oversimplified model may be accepted in normal market conditions, its reliability in times of distress has recently been questioned. In fact, counterparty risk has become one of the most dangerous threats to the CDS market. For some time now it has been realized that, in order to value a CDS properly, counterparty effects have to be taken into account (see ECB (2009)).

First, the value of CDS contracts tends to move very suddenly with big jumps, whereas the price movements of *IRS* contracts are far smoother and less volatile than *CDS* prices. Second, CDS spreads can widen very rapidly. Third, CDS contracts have many more risk factors than *IRS* contracts. In fact, our model shows that full collateralization *cannot* eliminate counterparty risk completely for a CDS contract.

The rest of this paper is organized as follows: Pricing multilateral defaultable financial contract is elaborated on in Section 2; numerical results are provided in Section 3; the conclusions are presented in Section 4. All proofs and some detailed derivations are contained in the appendices.

2 Pricing Financial Contracts Subject to Multilateral Credit Risk

We consider a filtered probability space (Ω , $\mathcal{F}$, $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{P}$) satisfying the usual conditions, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ -algebra, $\mathcal{P}$ denotes a probability measure, and $\{\mathcal{F}_t\}_{t\geq 0}$ denotes a filtration.

In the reduced-form approach, the stopping (or default) time τ_i of firm *i* is modeled as a Cox arrival process (also known as a doubly stochastic Poisson process) whose first jump occurs at default and is defined by,

$$\tau_i = \inf\left\{t : \int_0^t h_i(s, Z_s)ds \geq H_i\right\} \tag{1}$$

where $h_i(t)$ or $h_i(t, Z_t)$ denotes the stochastic hazard rate or arrival intensity dependent on an exogenous common state Z_t, and H_i is a unit exponential random variable independent of Z_t.

It is well-known that the survival probability from time *t* to *s* in this framework is defined by

$$p_i(t,s) := P_i(\tau > s \mid \tau > t, Z_t) = \exp\left(-\int_t^s h_i(u)du\right) \tag{2a}$$

The default probability for the period (*t, s*) in this framework is given by

$$q_i(t,s) := P_i(\tau \leq s \mid \tau > t, Z_t) = 1 - p_i(t,s) = 1 - \exp\left(-\int_t^s h_i(u)du\right) \tag{2b}$$

There is ample evidence that corporate defaults are correlated. The default of a firm's counterparty might affect its own default probability. Thus, default correlation/dependence occurs due to the counterparty relations.

The interest in the financial industry for the modeling and pricing of multilateral defaultable instruments arises mainly in two respects: in the management of credit risk at a portfolio level and in the valuation of credit derivatives. Central to the valuation and risk management of credit derivatives and risky portfolios is the problem of default relationship.

Let us discuss a three-party case first. A CDS is a good example of a trilateral defaultable instrument where the three parties are counterparties *A*, *B* and reference entity *C*. In a standard CDS contract one party purchases credit protection from another party, to cover the loss of the face value of a reference entity following a credit event. The protection buyer makes periodic payments to the seller until the maturity date or until a credit event occurs. A credit event usually requires a final accrual payment by the buyer and a loss protection payment by the protection seller. The protection payment is equal to the difference between par and the price of the cheapest to deliver (CTD) asset of the reference entity on the face value of the protection.

A CDS is normally used to transfer the credit risk of a reference entity between two counterparties. The contract reduces the credit risk of the reference entity but gives rise to another form of risk: *counterparty risk*. Since the dealers are highly concentrated within a small group, any of them may be too big to fail. The interconnected nature, with dealers being tied to each other through chains of OTC derivatives, results in increased contagion risk. Due to its concentration and interconnectedness, the CDS market seems to pose a systemic risk to financial market stability. In fact, the CDS is blamed for playing a pivotal role in the collapse of Lehman Brothers and the disintegration of AIG.

For years, a widespread practice in the market has been to mark CDS to market without taking the counterparty risk into account. The realization that even the most prestigious investment banks could go bankrupt has shattered the foundation of the practice. It is wiser to face frankly the real complexities of pricing a CDS than to indulge in simplifications that have proved treacherous. For some time now it has been realized that, in order to value a CDS properly, counterparty effects have to be taken into account.

Let *A* denote the protection buyer, *B* denote the protection seller and *C* denote the reference entity. The binomial default rule considers only two possible states: default or survival. Therefore, the default indicator Y_j for firm *j* (*j* = *A* or *B* or *C*) follows a Bernoulli distribution, which takes value 1 with default probability q_j, and value 0 with survival probability p_j. The marginal default distributions can be determined by the reduced-form models. The joint distributions of a multivariate Bernoulli variable can be easily obtained via the marginal distributions by introducing extra correlations. The joint probability representations of a trivariate Bernoulli distribution (see Teugels (1990)) are given by

$$p_{000} := P(Y_A = 0, Y_B = 0, Y_C = 0) = p_A p_B p_C + p_C \sigma_{AB} + p_B \sigma_{AC} + p_A \sigma_{BC} - \theta_{ABC} \tag{3a}$$

$$p_{100} := P(Y_A = 1, Y_B = 0, Y_C = 0) = q_A p_B p_C - p_C \sigma_{AB} - p_B \sigma_{AC} + q_A \sigma_{BC} + \theta_{ABC} \tag{3b}$$

$$p_{010} := P(Y_A = 0, Y_B = 1, Y_C = 0) = p_A q_B p_C - p_C \sigma_{AB} + q_B \sigma_{AC} - p_A \sigma_{BC} + \theta_{ABC} \tag{3c}$$

$$p_{001} := P(Y_A = 0, Y_B = 0, Y_C = 1) = p_A p_B q_C + q_C \sigma_{AB} - p_B \sigma_{AC} - p_A \sigma_{BC} + \theta_{ABC} \tag{3d}$$

$$p_{110} := P(Y_A = 1, Y_B = 1, Y_C = 0) = q_A q_B p_C + p_C \sigma_{AB} - q_B \sigma_{AC} - q_A \sigma_{BC} - \theta_{ABC} \tag{3e}$$

$$p_{101} := P(Y_A = 1, Y_B = 0, Y_C = 1) = q_A p_B q_C - q_C \sigma_{AB} + p_B \sigma_{AC} - q_A \sigma_{BC} - \theta_{ABC} \tag{3f}$$

$$p_{011} := P(Y_A = 0, Y_B = 1, Y_C = 1) = p_A q_B q_C - q_C \sigma_{AB} - q_B \sigma_{AC} + p_A \sigma_{BC} - \theta_{ABC} \tag{3g}$$

$$p_{111} := P(Y_A = 1, Y_B = 1, Y_C = 1) = q_A q_B q_C + q_C \sigma_{AB} + q_B \sigma_{AC} + q_A \sigma_{BC} + \theta_{ABC} \tag{3h}$$

where

$$\theta_{ABC} := E\big((Y_A - q_A)(Y_B - q_B)(Y_C - q_C)\big) \tag{3i}$$

Equation (3) tells us that the joint probability distribution of three defaultable parties depends not only on the bivariate statistical relationships of all pair-wise combinations (e.g., σ_{ij}) but also on the trivariate statistical relationship (e.g., θ_{ABC}). θ_{ABC} was first defined by Deardorff (1982) as *comvariance*, who use it to correlate three random variables that are the value of commodity net imports/exports, factor intensity, and factor abundance in international trading.

We introduce the concept of *comvariance* into credit risk modeling arena to exploit any statistical relationship among multiple random variables. Furthermore, we define a new statistic, *comrelation*, as a scaled version of comvariance (just like correlation is a scaled version of covariance) as follows:

Definition 1*: For three random variables* X_A, X_B, *and* X_C, *let* μ_A, μ_B, *and* μ_C *denote the means of* X_A, X_B, *and* X_C. *The comrelation of* X_A, X_B, *and* X_C *is defined by*

$$\zeta_{ABC} = \frac{E\big[(X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C)\big]}{\sqrt[3]{E|X_A - \mu_A|^3 \times E|X_B - \mu_B|^3 \times E|X_C - \mu_C|^3}} \tag{4}$$

According to the Holder inequality, we have

$$\begin{aligned} \big|E\big((X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C)\big)\big| &\le E\big|(X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C)\big| \\ &\le \sqrt[3]{E|X_A - \mu_A|^3 \times E|X_B - \mu_B|^3 \times E|X_C - \mu_C|^3} \end{aligned} \tag{5}$$

Obviously, the comrelation is in the range of [-1, 1]. Given the comrelation, Equation (3i) can be rewritten as

$$\begin{aligned}\theta_{ABC} &:= E\big((Y_A - q_A)(Y_B - q_B)(X_C - q_C)\big) = \zeta_{ABC}\sqrt[3]{E|Y_A - q_A|^3 \times E|Y_B - q_B|^3 \times E|Y_C - q_C|^3} \\ &= \zeta_{ABC}\sqrt[3]{\left|p_A q_A (p_A^2 + q_A^2) p_B q_B (p_B^2 + q_B^2) p_C q_C (p_C^2 + q_C^2)\right|}\end{aligned} \tag{6}$$

where $E(Y_j) = q_j$ and $E|Y_j - q_j|^3 = p_j q_j (p_j^2 + q_j^2)$, *j=A, B, or C.*

If we have a series of *n* measurements of X_A, X_B, *and* X_C written as x_{Ai}, x_{Bi} and x_{Ci} where *i* = 1,2,…,*n*, the sample *comrelation coefficient* can be obtained as:

$$\zeta_{ABC} = \frac{\sum_{i=1}^{n}(x_{Ai} - \mu_A)(x_{Bi} - \mu_B)(x_{Ci} - \mu_C)}{\sqrt[3]{\sum_{i=1}^{n}|x_{Ai} - \mu_A|^3 \times \sum_{i=1}^{n}|x_{Bi} - \mu_B|^3 \times \sum_{i=1}^{n}|x_{Ci} - \mu_C|^3}} \tag{7}$$

More generally, we define the *comrelation* in the context of *n* random variables as

***Definition 2**: For n random variables* X_1, X_2, ..., X_n, *let* μ_i *denote the mean of* X_i *where i=1,..,n. The comrelation of* X_1, X_2, ..., X_n *is defined as*

$$\zeta_{12\ldots n} = \frac{E\left[(X_1 - \mu_1)(X_2 - \mu_2)\cdots(X_n - \mu_n)\right]}{\sqrt[n]{E|X_1 - \mu_1|^n \times E|X_2 - \mu_2|^n \cdots \times E|X_n - \mu_n|^n}} \tag{8}$$

Correlation is just a specific case of comrelation where n = 2. Again, the comrelation $\zeta_{12\ldots n}$ is in the range of [-1, 1] according to the Holder inequality.

2.1 Risky valuation without collateralization

Recovery assumptions are important for pricing credit derivatives. If the reference entity under a CDS contract defaults, the best assumption, as pointed out by J. P. Morgan (1999), is that the recovered value equals the recovery rate times the face value plus accrued interest[1]. In other words, the recovery of

[1] In the market, there is an average accrual premium assumption, i.e., the average accrued premium is half the full premium due to be paid at the end of the premium.

par value assumption is a better fit upon the default of the reference entity, whereas the recovery of market value assumption is a more suitable choice in the event of a counterparty default[2].

Let valuation date be t. Suppose that a CDS has m scheduled payments represented as $X_i = -sN\delta(T_{i-1},T_i)$ with payment dates $T_1,\ldots,T_m$ where $i=1,,,,m$, $\delta(T_{i-1},T_i)$ denotes the accrual factor for period (T_{i-1},T_i), N denotes the notional/principal, and s denotes the CDS premium. Party A pays the premium/fee to party B if reference entity C does not default. In return, party B agrees to pay the protection amount to party A if reference entity C defaults before the maturity. We have the following proposition.

Proposition 1: *The value of the CDS is given by*

$$V(t) = \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} O(T_j,T_{j+1})\right)X_i \middle| \mathcal{F}_t\right] + \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-2} O(T_j,T_{j+1})\right)\Omega(T_{i-1},T_i)R(T_{i-1},T_i) \middle| \mathcal{F}_t\right] \tag{9a}$$

where $t = T_0$ *and*

$$O(T_j,T_{j+1}) = 1_{(V(T_{j+1})+X_{j+1})\geq 0}\,\phi_B(T_j,T_{j+1}) + 1_{(V(T_{j+1})+X_{j+1})<0}\,\phi_A(T_j,T_{j+1}) \tag{9b}$$

$$\begin{aligned}
\phi_A(T_j,T_{j+1}) = \{ & p_A(T_j,T_{j+1})p_B(T_j,T_{j+1})p_C(T_j,T_{j+1}) + q_A(T_j,T_{j+1})p_B(T_j,T_{j+1})p_C(T_j,T_{j+1})\varphi_A(T_{j+1}) \\
& + p_A(T_j,T_{j+1})q_B(T_j,T_{j+1})p_C(T_j,T_{j+1})\overline{\varphi}_A(T_{j+1}) + q_A(T_j,T_{j+1})q_B(T_j,T_{j+1})p_C(T_j,T_{j+1})\varphi_{AB}(T_{j+1}) \\
& + p_C(T_j,T_{j+1})\sigma_{AB}(T_j,T_{j+1})\left(1-\varphi_A(T_{j+1})-\overline{\varphi}_A(T_{j+1})+\varphi_{AB}(T_{j+1})\right) \\
& + \sigma_{AC}(T_j,T_{j+1})\left[p_B(T_j,T_{j+1})\left(1-\varphi_A(T_{j+1})\right)+q_B(T_j,T_{j+1})\left(\overline{\varphi}_A(T_{j+1})-\varphi_{AB}(T_{j+1})\right)\right] \\
& + \sigma_{BC}(T_j,T_{j+1})\left[p_A(T_j,T_{j+1})\left(1-\overline{\varphi}_A(T_{j+1})\right)+q_A(T_j,T_{j+1})\left(\varphi_A(T_{j+1})-\varphi_{AB}(T_{j+1})\right)\right] \\
& + \theta_{ABC}(T_j,T_{j+1})\left(-1+\overline{\varphi}_A(T_{j+1})-\varphi_{AB}(T_{j+1})+\varphi_A(T_{j+1})\right)\}D(T_j,T_{j+1})
\end{aligned} \tag{9c}$$

$$\begin{aligned}
\phi_B(T_j,T_{j+1}) = \{ & p_A(T_j,T_{j+1})p_B(T_J,T_{j+1})p_C(T_j,T_{j+1}) + q_A(T_j,T_{j+1})p_B(T_j,T_{j+1})p_C(T_j,T_{j+1})\overline{\varphi}_B(T_{j+1}) \\
& + p_A(T_j,T_{j+1})q_B(T_j,T_{j+1})p_C(T_j,T_{j+1})\varphi_B(T_{j+1}) + q_A(T_j,T_{j+1})q_B(T_j,T_{j+1})p_C(T_j,T_{j+1})\varphi_{AB}(T_{j+1}) \\
& + p_C(T_j,T_{j+1})\sigma_{AB}(T_j,T_{j+1})\left(1-\varphi_B(T_{j+1})-\overline{\varphi}_B(T_{j+1})+\varphi_{AB}(T_{j+1})\right) \\
& + \sigma_{AC}(T_j,T_{j+1})\left[p_B(T_j,T_{j+1})\left(1-\overline{\varphi}_B(T_{j+1})\right)+q_B(T_j,T_{j+1})\left(\varphi_B(T_{j+1})-\varphi_{AB}(T_{j+1})\right)\right] \\
& + \sigma_{BC}(T_j,T_{j+1})\left[p_A(T_j,T_{j+1})\left(1-\varphi_B(T_j)\right)+q_A(T_j,T_{j+1})\left(\overline{\varphi}_B(T_{j+1})-\varphi_{AB}(T_{j+1})\right)\right] \\
& + \theta(T_j,T_{j+1})\left(-1+\overline{\varphi}_B(T_{j+1})-\varphi_{AB}(T_{j+1})+\varphi_B(T_{j+1})\right)\}D(T_j,T_{j+1})
\end{aligned} \tag{9d}$$

[2] Three different recovery models exist in the literature. The default payoff is either i) a fraction of par (Madan and Unal (1998)), ii) a fraction of an equivalent default-free bond (Jarrow and Turnbull (1995)), or iii) a fraction of market value (Duffie and Singleton (1999)).

$$\begin{aligned}\Omega(T_j,T_{j+1}) = \{ & p_A(T_j,T_{j+1})p_B(T_j,T_{j+1})q_C(T_j,T_{j+1}) + q_A(T_j,T_{j+1})p_B(T_j,T_{j+1})q_C(T_j,T_{j+1})\overline{\varphi}_B(T_{j+1}) \\ & + p_A(T_j,T_{j+1})q_B(T_j,T_{j+1})q_C(T_j,T_{j+1})\varphi_B(T_{j+1}) + q_A(T_j,T_{j+1})q_B(T_j,T_{j+1})q_C(T_j,T_{j+1})\varphi_{AB}(T_{j+1}) \\ & + q_C(T_j,T_{j+1})\sigma_{AB}(T_j,T_{j+1})\big(1-\varphi_B(T_{j+1})-\overline{\varphi}_B(T_{j+1})+\varphi_{AB}(T_{j+1})\big) \\ & -\sigma_{AC}(T_j,T_{j+1})\big[p_B(T_j,T_{j+1})\big(1-\overline{\varphi}_B(T_{j+1})\big)+q_B(T_j,T_{j+1})\big(\varphi_B(T_{j+1})-\varphi_{AB}(T_{j+1})\big)\big] \\ & -\sigma_{BC}(T_j,T_{j+1})\big[p_A(T_j,T_{j+1})\big(1-\varphi_B(T_{j+1})\big)+q_A(T_j,T_{j+1})\big(\overline{\varphi}_B(T_{j+1})-\varphi_{AB}(T_{j+1})\big)\big] \\ & +\theta_{ABC}(T_j,T_{j+1})\big(1-\overline{\varphi}_B(T_{j+1})+\varphi_{AB}(T_{j+1})-\varphi_B(T_{j+1})\big)\}D(T_j,T_{j+1})\end{aligned} \tag{9e}$$

where $R(T_j,T_{j+1}) = \big(N\big(1-\varphi_C(T_{j+1})\big)-\alpha(T_j,T_{j+1})\big)$, $\alpha(T_j,T_{j+1}) = sN\delta(T_S,T)/2$, and $X_i = -sN\delta(T_j,T_{j+1})$.

Proof: See the Appendix.

We may think of $O(t,T)$ as the risk-adjusted discount factor for the premium and $\Omega(t,T)$ as the risk-adjusted discount factor for the default payment. Proposition 1 says that the pricing process of a multiple-payment instrument has a backward nature since there is no way of knowing which risk-adjusted discounting rate should be used without knowledge of the future value. Only on the maturity date, the value of an instrument and the decision strategy are clear. Therefore, the evaluation must be done in a backward fashion, working from the final payment date towards the present. This type of valuation process is referred to as backward induction.

Proposition 1 provides a general form for pricing a CDS. Applying it to a particular situation in which we assume that counterparties *A and B* are default-free, i.e., $p_j = 1$, $q_j = 0$, $\rho_{kl} = 0$, and $\varsigma_{ABC} = 0$, where *j=A or B* and *k, l=A, B, or C*, we derive the following corollary.

Corollary 1: *If counterparties A and B are default-free, the value of the CDS is given by*

$$\begin{aligned}V(t) &= \sum_{i=1}^m E\Big[\Big(\prod_{j=0}^{i-1} O(T_j,T_{j+1})\Big)X_i \big| \mathcal{F}_t\Big] + \sum_{i=1}^m E\Big[\Big(\prod_{j=0}^{i-2} O(T_j,T_{j+1})\Big)\Omega(T_{i-1},T_i)R(T_{i-1},T_i) \big| \mathcal{F}_t\Big] \\ &= \sum_{i=1}^m E\big[D(t,T_i)p_C(t,T_i)X_i \big| \mathcal{F}_t\big] + \sum_{i=1}^m E\big[D(t,T_i)p_C(t,T_{i-1})q_C(T_{i-1},T_i)R(T_{i-1},T_i) \big| \mathcal{F}_t\big]\end{aligned} \tag{10}$$

where $O(T_{i-1},T_i) = D(T_{i-1},T_i)p_C(T_{i-1},T_i)$; $\Omega(T_{i-1},T_i) = D(T_{i-1},T_i)q_C(T_{i-1},T_i)$.

The proof of this corollary becomes straightforward according to Proposition 1 by setting ρ_{kl} =0, $\varphi_{AB} = 0$, $\varsigma_{ABC} = 0$, $p_j = 1$, $q_j = 0$, $p_C(t,T_i) = \prod_{g=0}^{i-1} p(T_g,T_{g+1})$, and $D(t,T_i) = \prod_{g=0}^{i-1} D(T_g,T_{g+1})$.

If we further assume that the discount factor and the default probability of the reference entity are uncorrelated and the recovery rate φ_C is constant, we have

Corollary 2: *Assume that i) counterparties A and B are default-free, ii) the discount factor and the default probability of the reference entity are uncorrelated; iii) the recovery rate* φ_C *is constant; the value of the CDS is given by*

$$V(t) = \sum_{i=1}^{m} P(t,T_i)\bar{p}_C(t,T_{i-1})\bar{q}_C(T_{i-1},T_i)\left(N(1-\varphi_C) - \alpha(T_{i-1},T_i)\right) - \sum_{i=1}^{m} P(t,T_i)\bar{p}_c(t,T_i)sN\delta(T_{i-1},T_i) \tag{11}$$

where $P(t,T_i) = E[D(t,T_i)|\mathcal{F}_t]$ *denotes the bond price,* $\bar{p}_c(t,T_i) = E[p_c(t,T_i)|\mathcal{F}_t]$, $\bar{q}_c(t,T_i) = 1 - \bar{p}_c(t,T_i)$, $\bar{p}(t,T_{i-1})\bar{q}(T_{i-1},T_i) = \bar{p}(t,T_{i-1}) - \bar{p}(t,T_i)$.

This corollary is easily proved according to Corollary 1 by setting $E[XY|\mathcal{F}_t] = E[X|\mathcal{F}_t]E[Y|\mathcal{F}_t]$ when X and Y are uncorrelated. *Corollary 2 is the formula for pricing CDS in the market*.

Our methodology can be extended to the cases where the number of parties $n \geq 4$. A generating function for the (probability) joint distribution (see details in Teugels (1990)) of *n*-variate Bernoulli can be expressed as

$$p^{(n)} = \begin{bmatrix} p_n & -1 \\ q_n & 1 \end{bmatrix} \otimes \begin{bmatrix} p_{n-1} & -1 \\ q_{n-1} & 1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} p_1 & -1 \\ q_1 & 1 \end{bmatrix} \sigma^{(n)} \tag{12}$$

where $\otimes$ denotes the Kronecker product; $p^{(n)} = \{p_k^{(n)}\}$ and $\sigma^{(n)} = \{\sigma_k^{(n)}\}$ are vectors containing 2^n components: $p_k^{(n)} = p_{k_1,k_2,\ldots,k_n}$, $k = 1 + \sum_{i=1}^{n} k_i 2^{i-1}$, $k_i \in \{0,1\}$; $\sigma_k^{(n)} = \sigma_{k_1,k_2,\ldots,k_n} = E\left(\prod_{i=1}^{n}(Y_i - q_i)^{k_i}\right)$.

2.2 Risky valuation with collateralization

Collateralization is the most important and widely used technique in practice to mitigate credit risk. The posting of collateral is regulated by the Credit Support Annex (CSA) that specifies a variety of terms including the threshold, the independent amount, and the minimum transfer amount (MTA), etc. The threshold is the unsecured credit exposure that a party is willing to bear. The minimum transfer amount is the smallest amount of collateral that can be transferred. The independent amount plays the same role as the initial margin (or haircuts).

In a typical collateral procedure, a financial instrument is periodically marked-to-market and the collateral is adjusted to reflect changes in value. The collateral is called as soon as the mark-to-market

(MTM) value rises above the given collateral threshold, or more precisely, above the threshold amount plus the minimum transfer amount. Thus, the collateral amount posted at time t is given by

$$C(t) = \begin{cases} V(t) - H(t) & if\ V(t) > H(t) \\ 0 & otherwise \end{cases} \tag{13}$$

where $H(t)$ is the collateral threshold. In particular, $H(t)=0$ corresponds to full-collateralization[3]; $H>0$ represents partial/under-collateralization; and $H<0$ is associated with over-collateralization. Full collateralization becomes increasingly popular at the transaction level. In this paper, we focus on full collateralization only, i.e., $C(t)=V(t)$.

The main role of collateral should be viewed as an improved recovery in the event of a counterparty default. According to Bankruptcy law, if there has been no default, the collateral is returned to the collateral giver by the collateral taker. If a default occurs, the collateral taker possesses the collateral. In other words, collateral does not affect the survival payment; instead, it takes effect on the default payment only.

Collateral posting regimes are originally designed and utilized for bilateral risk products, e.g., IRS, but there are many reasons to be concerned about the success of collateral posting in offsetting the risks of CDS contracts. First, the values of CDS contracts tend to move very suddenly with big jumps, whereas the price movements of IRS contracts are far smoother and less volatile than CDS prices. Second, CDS spreads can widen very rapidly. The amount of collateral that one party is required to provide at short notice may, in some cases, be close to the notional amount of the CDS and may therefore exceed that party’s short-term liquidity capacity, thereby triggering a liquidity crisis. Third, CDS contracts have many more risk factors than IRS contracts.

[3] There are three types of collateralization: Full-collateralization is a process where the posting of collateral is equal to the current MTM value. Partial/under-collateralization is a process where the posting of collateral is less than the current MTM value. Over-collateralization is a process where the posting of collateral is greater than the current MTM value.

We assume that a CDS is fully collateralized, i.e., the posting of collateral is equal to the amount of the current *MTM* value: $C(t) = V(t)$. For a discrete one-period (*t*, *u*) economy, there are several possible states at time *u*: i) *A*, *B*, and *C* survive with probability p_{000}. The instrument value is equal to the market value $V(u)$; ii) *A* and *B* survive, but *C* defaults with probability p_{001}. The instrument value is the default payment $R(u)$; iii) For the remaining cases, either or both counterparties *A* and *B* default. The instrument value is the future value of the collateral $V(t)/D(t,u)$ (Here we consider the time value of money only). The value of the collateralized instrument at time *t* is the discounted expectation of all the payoffs and is given by

$$\begin{aligned} V(t) &= E\{D(t,u)[p_{000}(t,u)V(u) + p_{001}(t,u)R(u) \\ &\quad + \left(p_{100}(t,u) + p_{010}(t,u) + p_{110}(t,u) + p_{101}(t,u) + p_{011}(t,u) + p_{111}(t,u)\right)V(t)/D(t,u)] | \mathcal{F}_t\} \\ &= E\left\{\left[D(t,u)\left(p_{000}(t,u)V(s) + p_{001}(t,u)R(u)\right) + \left(1 - p_{000}(t,u) - p_{001}(t,u)\right)V(t)\right] | \mathcal{F}_t\right\} \end{aligned} \tag{14a}$$

or

$$\begin{aligned} &E\left[\left(p_A(t,u)p_B(t,u) + \sigma_{AB}(t,u)\right) | \mathcal{F}_t\right]V(t) \\ &\quad = E\{[D(t,u)\left(p_C(t,u)V(u) + q_C(t,u)R(u)\right)\left(p_A(t,u)p_B(t,u) + \sigma_{AB}(t,u)\right) \\ &\qquad + D(t,u)\left(V(u) - R(u)\right)\left(p_B(t,u)\sigma_{AC}(t,u) + p_A(t,u)\sigma_{BC}(t,u) - \theta_{ABC}(t,u)\right)] | \mathcal{F}_t\} \end{aligned} \tag{14b}$$

If we assume that $\left(p_A(t,u)p_B(t,u) + \sigma_{AB}(t,u)\right)$ and $D(t,u)\left(p_C(t,u)V(u) + q_C(t,u)R(u)\right)$ are uncorrelated, we have

$$V(t) = V^F(t) + \xi_{ABC}(t,u)/\psi_{AB}(t,u) \tag{15a}$$

where

$$V^F(t) = E\{D(t,u)[p_C(t,u)V(u) + q_C(t,u)R(u)] | \mathcal{F}_t\} \tag{15b}$$

$$\psi_{AB}(t,u) = E\left\{\left[p_A(t,u)p_B(t,u) + \sigma_{AB}(t,u)\right] | \mathcal{F}_t\right\} \tag{15c}$$

$$\xi_{ABC}(t,u) = E\left\{\left[D(t,u)\left(p_B(t,u)\sigma_{AC}(t,u) + p_A(t,u)\sigma_{BC}(t,u) - \theta_{ABC}(t,u)\right)\left(V(u) - R(u)\right)\right] | \mathcal{F}_t\right\} \tag{15d}$$

The first term $V^F(t)$ in equation (15) is the counterparty-risk-free value of the CDS and the second term is the exposure left over under full collateralization, which can be substantial.

Proposition 2*: If a CDS is fully collateralized, the risky value of the CDS is NOT equal to the counterparty-risk-free value, as shown in equation (15).*

Proposition 2 or equation (15) provides a theoretical explanation for the failure of full collateralization in the CDS market. It tells us that under full collateralization the risky value is in general not equal to the counterparty-risk-free value except in one of the following situations: i) the market value is equal to the default payment, i.e., $V(u) = R(u)$; ii) firms *A*, *B*, and *C* have independent credit risks, i.e., $\rho_{ij}=0$ and $\varsigma_{ABC} = 0$; or iii) the following relationship holds $p_B \sigma_{AC} + p_A \sigma_{BC} = \theta_{ABC}$.

4 Conclusion

To capture the default relationships among more than two defaultable entities, we introduce a new statistic: *comrelation*, an analogue to correlation for multiple variables, to exploit any multivariate statistical relationship. Our research shows that accounting for default correlations and comrelations becomes important, especially under market stress. The existing valuation models in the credit derivatives market, which take into account only pair-wise default correlations, may underestimate credit risk and may be inappropriate.

We study the sensitivity of the price of a defaultable instrument to changes in the joint credit quality of the parties. For instance, our analysis shows that the effect of default dependence on CDS premia from large to small is the correlation between the protection seller and the reference entity, the comrelation, the correlation between the protection buyer and the reference entity, and the correlation between the protection buyer and the protection seller.

The model shows that a fully collateralized CDS is not equivalent to a risk-free one. Therefore, we conclude that collateralization designed to mitigate counterparty risk works well for financial instruments subject to bilateral credit risk, but fails for ones subject to multilateral credit risk.

Appendix

Proof of Proposition 1. Let $t = T_0$. On the first payment date T_1, let $V(T_1)$ denote the market value of the CDS excluding the current cash flow X_1. There are a total of eight ($2^3 = 8$) possible states shown in Table A_1.

Table A_1. Payoffs of a trilaterally defaultable CDS

This table shows all possible payoffs at time T_1. In the case of $V(T_1) + X_1 \geq 0$ where $V(T_1)$ is the market value excluding the current cash flow X_1, there are a total of eight ($2^3 = 8$) possible states: i) *A*, *B*, and *C* survive with probability p_{000}. The instrument value equals the market value: $V(T_1) + X_1$. ii) *A* defaults, but *B* and *C* survive with probability p_{100}. The instrument value is a fraction of the market value: $\overline{\varphi}_B(T_1)(V(T_1) + X_1)$ where $\overline{\varphi}_B$ represents the non-default recovery rate of party *B*[4]. $\overline{\varphi}_B$ =0 represents the one-way settlement rule, while $\overline{\varphi}_B$ =1 represents the two-way settlement rule. iii) *A* and *C* survive, but *B* defaults with probability p_{010}. The instrument value is given by $\varphi_B(T_1)(V(T_1) + X_1)$ where φ_B represents the default recovery rate of defaulting party *B*. iv) *A* and *B* survive, but *C* defaults with probability p_{001}. The instrument value is the default payment: $R(T_0, T_1)$. v) *A* and *B* default, but *C* survives with probability p_{110}. The instrument value is given by $\varphi_{AB}(T_1)(V(T_1) + X_1)$ where φ_{AB} denotes the joint recovery rate when both parties *A* and *B* default simultaneously. vi) *A* and *C* default, but *B* survives with probability p_{101}. The instrument value is a fraction of the default payment: $\overline{\varphi}_B(T)R(T_0, T_1)$. vii) *B* and *C* default, but *A* survives with probability p_{011}, The instrument value is given by $\varphi_B(T)R(T_0, T_1)$. viii) *A*, *B*, and *C* default with

[4] There are two default settlement rules in the market. The *one-way payment rule* was specified by the early ISDA master agreement. The non-defaulting party is not obligated to compensate the defaulting party if the remaining market value of the instrument is positive for the defaulting party. The *two-way payment rule* is based on current ISDA documentation. The non-defaulting party will pay the full market value of the instrument to the defaulting party if the contract has positive value to the defaulting party.

probability p_{111}. The instrument value is given by $\varphi_{AB}(T)R(T_0,T_1)$. A similar logic applies to the case of $V(T_1)+X_1<0$.

Status	**Probability**	**Payoff if** $V(T_1)+X_1 \geq 0$	**Payoff if** $V(T_1)+X_1<0$
$Y_A=0, Y_B=0, Y_C=0$	p_{000}	$V(T_1)+X_1$	$V(T_1)+X_1$
$Y_A=1, Y_B=0, Y_C=0$	p_{100}	$\overline{\varphi}_B(T_1)(V(T_1)+X_1)$	$\varphi_A(T_1)(V(T_1)+X_1)$
$Y_A=0, Y_B=1, Y_C=0$	p_{010}	$\varphi_B(T_1)(V(T_1)+X_1)$	$\overline{\varphi}_A(T_1)(V(T_1)+X_1)$
$Y_A=0, Y_B=0, Y_C=1$	p_{001}	$R(T_0,T_1)$	$R(T_0,T_1)$
$Y_A=1, Y_B=1, Y_C=0$	p_{110}	$\varphi_{AB}(T_1)(V(T_1)+X_1)$	$\varphi_{AB}(T_1)(V(T_1)+X_1)$
$Y_A=1, Y_B=0, Y_C=1$	p_{101}	$\overline{\varphi}_B(T)R(T_0,T_1)$	$\overline{\varphi}_B(T)R(T_0,T_1)$
$Y_A=0, Y_B=1, Y_C=1$	p_{011}	$\varphi_B(T)R(T_0,T_1)$	$\varphi_B(T)R(T_0,T_1)$
$Y_A=1, Y_B=1, Y_C=1$	p_{111}	$\varphi_{AB}(T)R(T_0,T_1)$	$\varphi_{AB}(T)R(T_0,T_1)$

The risky price is the discounted expectation of the payoffs and is given by

$$
\begin{aligned}
V(t) = E\Big\{\Big[&1_{(V(T_1)X_1\geq 0}\big\langle\big(p_{000}(T_0,T_1)+p_{100}(T_0,T_1)\overline{\varphi}_B(T_1)+p_{010}(T_0,T_1)\varphi_B(T_1)+p_{110}(T_0,T_1)\varphi_{AB}(T_1)\big)(V(T_1)+X_1)\big|\mathcal{F}_t\big\rangle \\
&+1_{(V(T_1)+X_1)<0}\big\langle\big(p_{000}(T_0,T_1)+p_{100}(T_0,T_1)\varphi_A(T_1)+p_{010}(T_0,T_1)\overline{\varphi}_A(T_1)+p_{110}(T_0,T_1)\varphi_{AB}(T_1)\big)(V(T_1)+X_1)\big|\mathcal{F}_t\big\rangle \\
&+\big(p_{001}(T_0,T_1)+p_{101}(T_0,T_1)\overline{\varphi}_B(T_1)+p_{011}(T_0,T_1)\varphi_B(T_1)+p_{111}(T_0,T_1)\varphi_{AB}(T_1)\big)R(T_s,T_1)\big|\mathcal{F}_t\Big]D(t,T)\Big\} \\
= E\big\{&\big[O(T_0,T_1)(V(T_1)+X_1)+\Omega(T_0,T_1)R(T_s,T_1)\big]\big|\mathcal{F}_t\big\}
\end{aligned}
\tag{A1a}
$$

where

$$
O(T_0,T_1)=1_{(V(T_1)+X_1)\geq 0}\phi_B(T_0,T_1)+1_{(V(T_1)+X_1)<0}\phi_A(T_0,T_1) \tag{A1b}
$$

$$
\begin{aligned}
\phi_A(T_0,T_1)=\big\{&p_A(T_0,T_1)p_B(T_0,T_1)p_C(T_0,T_1)+q_A(T_0,T_1)p_B(T_0,T_1)p_C(T_0,T_1)\varphi_A(T_1) \\
&+p_A(T_0,T_1)q_B(T_0,T_1)p_C(T_0,T_1)\overline{\varphi}_A(T_1)+q_A(T_0,T_1)q_B(T_0,T_1)p_C(T_0,T_1)\varphi_{AB}(T_1) \\
&+p_C(T_0,T_1)\sigma_{AB}(T_0,T_1)\big(1-\varphi_A(T_1)-\overline{\varphi}_A(T_1)+\varphi_{AB}(T_1)\big) \\
&+\sigma_{AC}(T_0,T_1)\big[p_B(T_0,T_1)\big(1-\varphi_A(T_1)\big)+q_B(T_0,T_1)\big(\overline{\varphi}_A(T_1)-\varphi_{AB}(T_1)\big)\big] \\
&+\sigma_{BC}(T_0,T_1)\big[p_A(T_0,T_1)\big(1-\overline{\varphi}_A(T_1)\big)+q_A(T_0,T_1)\big(\varphi_A(T_1)-\varphi_{AB}(T_1)\big)\big] \\
&+\theta_{ABC}(T_0,T_1)\big(-1+\overline{\varphi}_A(T_1)-\varphi_{AB}(T_1)+\varphi_A(T_1)\big)\big\}D(T_0,T_1)
\end{aligned}
\tag{A1c}
$$

$$\begin{aligned}\phi_B(T_0,T_1) = \{ & p_A(T_0,T_1)p_B(T_0,T_1)p_C(T_0,T_1) + q_A(T_0,T_1)p_B(T_0,T_1)p_C(T_0,T_1)\overline{\varphi}_B(T_1) \\ & + p_A(T_1,T_1)q_B(T_0,T_1)p_C(T_0,T_1)\varphi_B(T_1) + q_A(T_0,T_1)q_B(T_0,T_1)p_C(T_0,T_1)\varphi_{AB}(T_1) \\ & + p_C(T_0,T_1)\sigma_{AB}(T_0,T_1)\left(1-\varphi_B(T_1)-\overline{\varphi}_B(T_1)+\varphi_{AB}(T_1)\right) \\ & + \sigma_{AC}(T_0,T_1)\left[p_B(T_0,T_1)\left(1-\overline{\varphi}_B(T_1)\right)+q_B(T_0,T_1)\left(\varphi_B(T_1)-\varphi_{AB}(T_1)\right)\right] \\ & + \sigma_{BC}(T_0,T_1)\left[p_A(T_0,T_1)\left(1-\varphi_B(T_1)\right)+q_A(T_0,T_1)\left(\overline{\varphi}_B(T_1)-\varphi_{AB}(T_1)\right)\right] \\ & + \theta_{ABC}(T_0,T_1)\left(-1+\overline{\varphi}_B(T_1)-\varphi_{AB}(T_1)+\varphi_B(T_1)\right)\}D(T_0,T_1)\end{aligned} \tag{A1d}$$

$$\begin{aligned}\Omega(T_0,T) = \{ & p_A(T_0,T)p_B(T_0,T)q_C(T_0,T) + q_A(T_0,T)p_B(T_0,T)q_C(T_0,T)\overline{\varphi}_B(T_1) \\ & + p_A(T_0,T)q_B(T_0,T)q_C(T_0,T)\varphi_B(T_1) + q_A(T_0,T)q_B(T_0,T)q_C(T_0,T)\varphi_{AB}(T_1) \\ & + q_C(T_0,T)\sigma_{AB}(T_0,T)\left(1-\varphi_B(T_1)-\overline{\varphi}_B(T_1)+\varphi_{AB}(T_1)\right) \\ & - \sigma_{AC}(T_0,T)\left[p_B(T_0,T)\left(1-\overline{\varphi}_B(T_1)\right)+q_B(T_0,T)\left(\varphi_B(T_1)-\varphi_{AB}(T_1)\right)\right] \\ & - \sigma_{BC}(T_0,T)\left[p_A(T_0,T)\left(1-\varphi_B(T_1)\right)+q_A(T_0,T)\left(\overline{\varphi}_B(T_1)-\varphi_{AB}(T_1)\right)\right] \\ & + \theta_{ABC}(T_0,T)\left(1-\overline{\varphi}_B(T_1)+\varphi_{AB}(T_1)-\varphi_B(T_1)\right)\}D(T_0,T)\end{aligned} \tag{A1e}$$

Similarly, we have

$$V(T_1) = E\left\{\left[O(T_1,T_2)\left(X_2+V(T_2)\right)+\Omega(T_1,T_2)R(T_1,T_2)\right]\Big|\mathcal{F}_{T_1}\right\} \tag{A2}$$

Note that $O(T_0,T_1)$ is $\mathcal{F}_{T_1}$ -measurable. By definition, an $\mathcal{F}_{T_1}$ -measurable random variable is a random variable whose value is known at time T_1. According to *taking out what is known* and *towe*r properties of conditional expectation, we have

$$\begin{aligned}V(t) &= E\left\{\left[O(T_0,T_1)\left(X_1+V(T_1)\right)+\Omega(T_0,T_1)R(T_0,T_1)\right]\big|\mathcal{F}_t\right\} = E\left[O(T_0,T_1)X_1\big|\mathcal{F}_t\right] \\ &\quad + E\left[\Omega(T_0,T_1)R(T_0,T_1)\big|\mathcal{F}_t\right] + E\left\{O(T_0,T_1)E\left\langle\left[O(T_1,T_2)\left(X_2+V(T_2)\right)+\Omega(T_1,T_2)R(T_1,T_2)\right]\Big|\mathcal{F}_{T_1}\right\rangle\big|\mathcal{F}_t\right\} \\ &= \sum_{i=1}^{2} E\left[\left(\prod_{j=0}^{i-1} O(T_j,T_{j+1})\right)X_i\right] + \sum_{i=1}^{2} E\left[\left(\prod_{j=0}^{i-2} O(T_j,T_{j+1})\right)\Omega(T_{i-1},T_i)R(T_{i-1},T_i)\right] \\ &\quad + E\left[\left(\prod_{j=0}^{1} O(T_j,T_{j+1})\right)V(T_2)\big|\mathcal{F}_t\right]\end{aligned} \tag{A3}$$

By recursively deriving from T_2 forward over T_m, where $V(T_m) = X_m$, we have

$$V(t) = \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} O(T_j,T_{j+1})\right)X_i\big|\mathcal{F}_t\right] + \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-2} O(T_j,T_{j+1})\right)\Omega(T_{i-1},T_i)R(T_{i-1},T_i)\big|\mathcal{F}_t\right] \tag{A4}$$

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