Optimal decay rates of a nonlinear suspension bridge with memories

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Abstract

In this paper, we investigate the decay properties of suspension bridge with memories in one dimension. To prove our results, we use the energy method to build some very delicate Lyapunov functionals that give the desired results.

Keywords: Suspension bridges, memory term, energy method, Lyapunov functional

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1 Introduction

Many suspension bridges manifested aerodynamic instability and uncontrolled oscillations leading to collapses, see ([2], [8], [21], [26], [27], [31]). These accidents are due to several different causes like the one mentioned in caused [4] by torsional oscillations.

The instability of suspension bridges raised some fundamental questions for both engineers and mathematicians. To explain such instability issues in the suspension bridges many models were introduced in the literature see ([3], [16], [6], [4], [19]). Our objective in this paper is to get the stability of the suspension bridge model by introducing a memory damping.

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Mainly, our interest is the get the behavior of the solution of the following coupled nonlinear system with memories in one dimensional space:

$$\begin{cases} u_{tt}+u_{xxxx}-\int_0^t g_1(t-s)\,u_{xxxx}(s)\,ds+\left(\int_0^l\left(u^2+\theta^2\right)dx\right)u+2\left(\int_0^l u\theta dx\right)\theta=0, & in\ (0,l)\times\mathbb{R}_+, \\ \theta_{tt}-\theta_{xx}+\int_0^t g_2(t-s)\,\theta_{xx}(s)\,ds+2\left(\int_0^l u\theta dx\right)u+\left(\int_0^l\left(u^2+\theta^2\right)dx\right)\theta=0, & in\ (0,l)\times\mathbb{R}_+, \end{cases} \tag{1.1}$$

with the initial data

$$(u,u_t,\theta,\theta_t)(x,0)=(u_0,u_1,\theta_0,\theta_1), \tag{1.2}$$

Since the plate is hinged between the two towers and the cross sections between the towers cannot rotate, the boundary conditions to be associated

$$u(0,t)=u(l,t)=u_{xx}(0,t)=u_{xx}(l,t)=\theta(0,t)=\theta(l,t)=0;\ \forall t\geq 0. \tag{1.3}$$

The given system models a one-dimensional suspension bridge, where u is the vertical displacement, θ is the torsional angle, The integral term represents a history terms with kernel g_i, for $i=1,2$ and satisfying the following hypotheses:

(A_1) $g_i:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ is a nonincreasing C^1 function satisfying.

$$g_i(0)>0;\ 1-\int_0^\infty g_i(s)\,ds=l_i>0.$$

(A_2) There exists a C^1 function $H_i:\mathbb{R}_+\longrightarrow\mathbb{R}_+$; which is linear or it is an strictly increasing and strictly convex function of class C^2 on $(0,r]$, $r\leq g_i(0)$,with $H_i(0)=H_i'(0)=0$ such that,

$$\begin{cases} g_i'(t)\leq -\xi_i(t)\ H_i(g_i(t))\,; \quad \forall t\geq 0, \\ \text{where } \xi_i(t) \text{ is a } C^1 \text{ function satisfying: } \xi_i(t)>0 \text{ and } \xi_i'(t)\leq 0; \quad \forall t\geq 0. \end{cases}$$

The analysis and the stability of various nonlinear suspension bridge models has been attracted many researchers (see: [1], [13], [9], [10], [18], [22]) and the references therein. For the suspension bridge models with viscoelastic or with memory, we can mention the works of [11] where the authors proved the asymptotic behavior of the solutions of the viscoelastic suspension bridge model, Kang in [23] where he proved the long-time behavior to the suspension bridge equation under the memory term. Also, Kang in [24] provide a result on the global attractor to a thermoelastic suspension bridge equation with past history, and recently Mukiawa in [28] proved the asymptotic behavior of the solutions of the suspension bridges with viscoelastic damping.

In this paper, we establish a general decay result under the conditions $(A1-A2)$ on the relaxation functions g_1 and g_2. Our proof is based on the multipliers techniques and the construction of the appropriate Lyapunov functional.

We point out here that our argument is close to the one in ([14]), with the necessary modifications required by the nature of our model.

The outline of this paper is as follow: In Section 2, we state and prove the general stability result for system (1.1). In Section 3, we give conclusion and open question.

2 General stability result

In this section, we state and prove a general stability result for (1.1) with boundary and initial conditions given by (1.3) and (1.2).

We state without proof a global existence result. Throughout this paper, c is used to denote a generic positive constant.

The well-posedness of (1.1) is stated in the following proposition.

Theorem 2.1. *Supposing that (A1)-(A2) hold and that* $((u_0, u_1), (\theta_0, \theta_1)) \in ((H_0^2(0,l) \times H_0^1(0,l)) \times (H_0^1(0,l) \times L^2(0,l))$*, then there exits* $T > 0$ *and a unique solution* $(u, \theta)(t)$ *of problem (1.1) such that*

$$(u, \theta)(t) \in C\left([0,T]\,;\, H_0^2(0,l) \times H_0^1(0,l)\right) \cap C^1\left([0,T]\,;\, H_0^1(0,l) \times L^2(0,l)\right).$$

Proof. This theorem can be established using standard method such as Galerkin method. □

The first-order energy associated with (1.1) is given by:

$$E(t) = \frac{1}{2}\left(\begin{array}{c} \int_0^l \left(u_t^2 + \theta_t^2 + \left(1 - \int_0^t g_1(s)\,ds\right) u_{xx}^2 + \left(1 - \int_0^t g_2(s)\,ds\right)\theta_x^2\right) dx \\ +g_1 \circ u_{xx}(t) + g_2 \circ \theta_x(t) + \left(\int_0^l \left(u^2 + \theta^2\right) dx\right)^2 + 2\left(\int_0^l u\theta\, dx\right)^2 \end{array}\right), \tag{2.1}$$

where

$$g \circ \varphi(t) = \int_0^t g(t-s)\, \|\varphi(t) - \varphi(s)\|_{L^2(0,l)}^2\, ds.$$

Our objective is to prove the general decay using the energy method.

Theorem 2.2. *Let* $((u_0, u_1), (\theta_0, \theta_1)) \in ((H_0^2(0,1) \times H_0^1(0,1)) \times (H_0^1(0,l) \times L^2(0,l))$ *be given and assume that (A1) and (A2) are satisfied. Then, there exist two positive constants* C_0 *and* $C_1 \preceq 1$ *such that*

$$E(t) \le C_0\, H_4^{-1}\left(C_1 \int_{t_0}^t \xi(s)\, ds\right); \qquad \forall t \ge t_0, \tag{2.2}$$

where

$$H_4(t) = \int_t^r \frac{1}{s\, H_0(s)} ds, \quad H_0(t) = \min\{H_1'(t), H_2'(t)\},$$

and $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$, $t_0 = \max(t_1, t_2)$ *with* $g_1(t_1) = r$ *and* $g_2(t_2) = r$.

The proof of this Theorem will be established through several lemmas.

Here, we will deduce some remarks.

Remark 2.1. *1. From (A_1) we infer that $\lim_{t\to+\infty} g_i(t) = 0$; $i \in \{1,2\}$. Then there exists some $t_i \geq 0$ large enough such that*

$$g_i(t_i) \leq r \Longrightarrow g_i(t) \leq r, \qquad \forall t \geq t_i. \tag{2.3}$$

Since H_i is a positive continuous function and g_i, ξ_i are positive nonincreasing continuous functions, with $t_0 = \max(t_1, t_2)$, we can get for every $t \in [0, t_0]$

$$0 < g_i(t_0) \leq g_i(t) \leq g_i(0) \text{ and } 0 < \xi_i(t_0) \leq \xi_i(t) \leq \xi_i(0),$$

which implies for some positive constants a_i and b_i,

$$a_i \leq \xi_i(t)\ H_i(g_i(t)) \leq b_i,$$

this shows that for every $t \in [0, t_0]$,

$$g_i'(t) \leq -\xi_i(t)\ H_i(g_i(t)) \leq -\frac{a_i}{g_i(0)} g_i(0) \leq -\frac{a_i}{g_i(0)} g_i(t). \tag{2.4}$$

2. If (2.2) is given

$$g_i'(t) \leq -\xi_i(t)\ g_i^p(t), \qquad 1 \leq p < 12\ i = 1,2.$$

Then there exist positive constants k, k_1 and k_2 such that

$$E(t) \preceq \begin{cases} k \exp\left(-k_1 \int_0^t \xi(s)\, \xi_2(s)\ ds\right), & if\ p = 1, \\ k_2 \left(1 + \int_0^t \xi(s)\ ds\right)^{-\dfrac{1}{p-1}}, & if\ 1 < p < 2. \end{cases}$$

Lemma 2.3. *Let (u, θ) be the solution of (1.1), then the energy functional E, defined by (2.1) satisfies*

$$E'(t) = -\frac{1}{2} g_1(t) \|u_{xx}\|^2_{L^2(0,l)} - \frac{1}{2} g_2(t) \|\theta_x\|^2_{L^2(0,l)} + \frac{1}{2} g_1' \circ u_{xx}(t) + \frac{1}{2} g_2' \circ \theta_x(t). \tag{2.5}$$

Proof. Multiplying the first equation of (1.1) by u_t , the second by θ_t, integrating by parts over $(0, l)$ and using the boundary conditions (1.3), then summing up, we obtain the result. $\square$

Lemma 2.4. *Let (u, θ) be the solution of (1.1). Then the functional*

$$I_1(t) = \int_0^l u\, u_t\, dx + \int_0^l \theta\, \theta_t\, dx,$$

satisfies, for all $\varepsilon_1 > 0$, the following estimate

$$\begin{aligned} I_1'(t) &\leq \|u_t\|^2_{L^2(0,l)} + \|\theta_t\|^2_{L^2(0,l)} - (1-\varepsilon_1)\|u_{xx}\|^2_{L^2(0,l)} - (1-\varepsilon_1)\|\theta_x\|^2_{L^2(0,l)} \\ &\quad - \left(\int_0^l \left(u^2+\theta^2\right)dx\right)^2 - 4\left(\int_0^l u\theta dx\right)^2 \\ &\quad + \frac{C_{\alpha_1}}{\varepsilon_1}\left(h_1 \circ u_{xx}(t)\right) + \frac{C_{\alpha_2}}{\varepsilon_1}\left(h_2 \circ \theta_x(t)\right), \end{aligned} \tag{2.6}$$

for any $0 < \alpha_1 < 1$ and $0 < \alpha_2 < 1$ where

$$\begin{aligned} C_{\alpha_1} &= \int_0^\infty \frac{g_1^2(s)}{\alpha_1 g_1(s) - g_1'(s)}ds, \\ C_{\alpha_2} &= \int_0^\infty \frac{g_2^2(s)}{\alpha_2 g_2(s) - g_2'(s)}ds, \end{aligned} \quad \textit{and} \quad \begin{aligned} h_1(t) &= \left(\alpha_1 g_1(t) - g_1'(t)\right), \\ h_2(t) &= \left(\alpha_2 g_2(t) - g_2'(t)\right). \end{aligned}$$

Proof. Using $(1.1)_1$, $(1.1)_2$, integrating by parts over $(0,l)$ and using the boundary conditions (1.3) , we obtain

$$\begin{aligned} I_1'(t) &= \|u_t\|^2_{L^2(0,l)} + \|\theta_t\|^2_{L^2(0,l)} - \|u_{xx}\|^2_{L^2(0,l)} - \|\theta_x\|^2_{L^2(0,l)} \\ &\quad - \left(\int_0^l \left(u^2+\theta^2\right)dx\right)^2 - 4\left(\int_0^l u\theta dx\right)^2 \\ &\quad + \int_0^l u_{xx}(t)\int_0^t g_1(t-s)\left(u_{xx}(t) - u_{xx}(s)\right)ds \\ &\quad + \int_0^l \theta_x(t)\int_0^t g_2(t-s)\left(\theta_x(t) - \theta_x(s)\right)ds\,. \end{aligned} \tag{2.7}$$

First by Hölder's inequality, we get

$$\begin{aligned} &\int_0^l \left(\int_0^t g_1(t-s)\left|u_{xx}(t) - u_{xx}(s)\right|ds\right)^2 dx \\ &= \int_0^l \left(\int_0^t \frac{g_1(t-s)}{\sqrt{\alpha_1 g_1(s) - g_1'(s)}}\sqrt{\alpha_1 g_1(s) - g_1'(s)}\left|u_{xx}(t) - u_{xx}(s)\right|ds\right)^2 dx \\ &\leq \int_0^t \frac{g_1^2(s)}{\alpha_1 g_1(s) - g_1'(s)}\int_0^l\int_0^t \left[\alpha_1 g_1(s) - g_1'(s)\right]\left|u_{xx}(t) - u_{xx}(s)\right|^2 dsdx \\ &= C_{\alpha_1}\left(h_1 \circ u_{xx}(t)\right), \end{aligned} \tag{2.8}$$

also, with the same approach, we get

$$\int_0^l \left(\int_0^t g_2(t-s)\left|\theta_x(t) - \theta_x(s)\right|ds\right)^2 dx \leq C_{\alpha_2}\left(h_2 \circ \theta_x(t)\right). \tag{2.9}$$

By exploiting the properties of g_i, and using Cauchy–Schwarz and Young's inequalities and

(2.8), we obtain, for any $\varepsilon_1 > 0$

$$\begin{aligned}
&\int_0^l u_{xx}(t) \int_0^t g_1(t-s)\,(u_{xx}(t)-u_{xx}(s))\,ds \qquad (2.10)\\
\leq\;& \varepsilon_1 \|u_{xx}\|^2_{L^2(0,l)} + \frac{1}{4\varepsilon_1}\int_0^l \left[\int_0^t g_1(t-s)\,|u_{xx}(t)-u_{xx}(s)|\,ds\right]^2\\
\leq\;& \varepsilon_1 \|u_{xx}\|^2_{L^2(0,l)} + \frac{C_{\alpha_1}}{\varepsilon_1}(h_1 \circ u_{xx}(t)),
\end{aligned}$$

also, with the same approach and by using (2.9), we get

$$\int_0^l \theta_x(t)\int_0^t g_2(t-s)\,(\theta_x(t)-\theta_x(s))\,dsdx \;\leq\; \varepsilon_1\|\theta_x\|^2_{L^2(0,l)} + \frac{C_{\alpha_2}}{\varepsilon_1}(h_2\circ\theta_x(t)), \qquad (2.11)$$

by (2.7), (2.10) and (2.11), we deduce the result. □

Lemma 2.5. *Let (u,θ) be the solution of (1.1). Then the functional*

$$\begin{aligned}
I_2 \;=\; & -\int_0^l u_t(t)\int_0^t g_1(t-s)\,(u(t)-u(s))\,dsdx\\
& -\int_0^l \theta_t(t)\int_0^t g_2(t-s)\,(\theta(t)-\theta(s))\,dsdx.
\end{aligned}$$

satisfies, for all $\varepsilon_3,\varepsilon_5,\varepsilon_7,\varepsilon_9>0$, the following estimate

$$I_2'(t) \leq -\left(\int_0^t g_1(s)\,ds - \varepsilon_9\right)\|u_t\|^2_{L^2(0,l)} - \left(\int_0^t g_2(s)\,ds-\varepsilon_9\right)\|\theta_t\|^2_{L^2(0,l)} \qquad (2.12)$$

$$\begin{aligned}
&+\varepsilon_7\sqrt{E(0)}\left(\int_0^l u\theta dx\right)^2 + \varepsilon_7\sqrt{E(0)}\left(\int_0^l (u^2+\theta^2)\,dx\right)^2\\
&+\left(\left(1+\frac{2C_p}{\varepsilon_7}+\frac{[1-l_1]^2}{\varepsilon_5}+\frac{1}{\varepsilon_3}\right)C_{\alpha_1} + \frac{C_p}{2\varepsilon_9}\left(\alpha_1(1-l_1)+g_1(0)+\alpha_1^2 C_{\alpha_1}\right)\right)(h_1\circ u_{xx}(t))\\
&+\left(\left(1+\frac{2C_p}{\varepsilon_7}+\frac{[1-l_2]^2}{\varepsilon_5}+\frac{1}{\varepsilon_3}\right)C_{\alpha_2} + \frac{C_p}{2\varepsilon_9}\left(\alpha_2(1-l_2)+g_2(0)+\alpha_2^2 C_{\alpha_2}\right)\right)(h_2\circ\theta_x(t))\\
&+(\varepsilon_3+\varepsilon_5)\|u_{xx}\|^2_{L^2(0,l)} + (\varepsilon_3+\varepsilon_5)\|\theta_x\|^2_{L^2(0,l)}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
I_2'(t) \;=\; & -\int_0^l u_{tt}(t)\int_0^t g_1(t-s)\,(u(t)-u(s))\,dsdx - \int_0^l u_t(t)\int_0^t g_1'(t-s)\,(u(t)-u(s))\,dsdx\\
& -\left(\int_0^t g_1(s)\,ds\right)\|u_t\|^2_{L^2(0,l)} - \left(\int_0^t g_2(s)\,ds\right)\|\theta_t\|^2_{L^2(0,l)} \qquad (2.13)\\
& -\int_0^l \theta_{tt}(t)\int_0^t g_2(t-s)\,(\theta(t)-\theta(s))\,dsdx - \int_0^l \theta_t(t)\int_0^t g_2'(t-s)\,(\theta(t)-\theta(s))\,dsdx.
\end{aligned}$$

By using $(1.1)_1$, $(1.1)_2$, integrating by parts over $(0,l)$ and using the boundary conditions (1.3) , we deduce from (2.13)

$$
\begin{aligned}
I_2'(t) &= \int_0^l u_{xx}(t)\int_0^t g_1(t-s)\,(u_{xx}(t)-u_{xx}(s))\,dsdx \qquad (2.14)\\
&-\int_0^l\int_0^t g_1(t-s)\,u_{xx}(s)\,ds\int_0^t g_1(t-s)\,(u_{xx}(t)-u_{xx}(s))\,dsdx\\
&+\int_0^l\left(\left(\int_0^l\left(u^2+\theta^2\right)dx\right)u(t)\right)\int_0^t g_1(t-s)\,(u(t)-u(s))\,dsdx\\
&+\int_0^l\left(2\left(\int_0^l u\theta dx\right)\theta\ (t)\right)\int_0^t g_1(t-s)\,(u(t)-u(s))\,dsdx\\
&-\int_0^l u_t(t)\int_0^t g_1'(t-s)\,(u(t)-u(s))\,dsdx-\left(\int_0^t g_1(s)\,ds\right)\|u_t\|^2_{L^2(0,l)}\\
&+\int_0^l \theta_x(t)\int_0^t g_2(t-s)\,(\theta_x(t)-\theta_x(s))\,dsdx\\
&-\int_0^l\int_0^t g_2(t-s)\,\theta_x(s)\,ds\int_0^t g_2(t-s)\,(\theta_x(t)-\theta_x(s))\,dsdx\\
&+\int_0^l\left(\left(\int_0^l\left(u^2+\theta^2\right)dx\right)\theta(t)\right)\int_0^t g_2(t-s)\,(\theta(t)-\theta(s))\,dsdx\\
&+\int_0^l\left(2\left(\int_0^l u\theta dx\right)u\ (t)\right)\int_0^t g_2(t-s)\,(\theta(t)-\theta(s))\,dsdx\\
&-\int_0^l \theta_t(t)\int_0^t g_2'(t-s)\,(\theta(t)-\theta(s))\,dsdx-\left(\int_0^t g_2(s)\,ds\right)\|\theta_t\|^2_{L^2(0,l)}
\end{aligned}
$$

1. As in (2.10) and (2.11), we have for any $\varepsilon_3>0$

$$\int_0^l u_{xx}(t)\int_0^t g_1(t-s)\,(u_{xx}(t)-u_{xx}(s))\,dsdx\le\varepsilon_3\|u_{xx}\|^2_{L^2(0,l)}+\frac{C_{\alpha_1}}{\varepsilon_3}\left(h_1\circ u_{xx}(t)\right), \tag{2.15}$$

and

$$\int_0^l \theta_x(t)\int_0^t g_2(t-s)\,(\theta_x(t)-\theta_x(s))\,dsdx\le\varepsilon_3\|\theta_x\|^2_{L^2(0,l)}+\frac{C_{\alpha_2}}{\varepsilon_3}\left(h_2\circ\theta_x(t)\right). \tag{2.16}$$

By exploiting the properties of g_i, and using Cauchy–Schwarz and Young's inequalities and the fact that $\left(1-\int_0^\infty g_i(s)\,ds\right)=l_i;\ i\in\{1,2\}$ and (2.8) and (2.15), we obtain, for any $\varepsilon_5 > 0$

$$\begin{aligned}
& -\int_0^l \int_0^t g_1(t-s)\, u_{xx}(s)\, ds \int_0^t g_1(t-s)\, \left(u_{xx}(t) - u_{xx}(s)\right) ds dx \qquad (2.17) \\
= & \int_0^l \left[\int_0^t g_1(t-s)\, \left(u_{xx}(t) - u_{xx}(s)\right)\right]^2 ds dx \\
& -\left[\int_0^t g_1(t-s)\, ds\right] \int_0^l u_{xx}(t) \int_0^t g_1(t-s)\, \left(u_{xx}(t) - u_{xx}(s)\right) ds dx \\
\leq & \ \varepsilon_5 \left\| u_{xx} \right\|_{L^2(0,l)}^2 + \left(\frac{[1-l_1]^2 C_{\alpha_1}}{\varepsilon_5} + C_{\alpha_1}\right) \left(h_1 \circ u_{xx}(t)\right),
\end{aligned}$$

and with the same approach and (2.9) and (2.16), we have

$$\begin{aligned}
& -\int_0^l \int_0^t g_2(t-s)\, \theta_x(s)\, ds \int_0^t g_2(t-s)\, \left(\theta_x(t) - \theta_x(s)\right) ds dx \qquad (2.18) \\
\leq & \ \varepsilon_5 \left\| \theta_x \right\|_{L^2(0,l)}^2 + \left(\frac{[1-l_2]^2 C_{\alpha_2}}{\varepsilon_5} + C_{\alpha_2}\right) \left(h_2 \circ \theta_x(t)\right).
\end{aligned}$$

Using Cauchy–Schwarz, Young and Poincaré's inequalities and (2.8), for any $\varepsilon_7 > 0$

$$\begin{aligned}
& \int_0^l \left(\left(\int_0^l \left(u^2 + \theta^2\right) dx\right) u(t)\right) \int_0^t g_1(t-s)\, \left(u(t) - u(s)\right) ds dx \qquad (2.19) \\
\leq & \ \varepsilon_7 \left(\int_0^l \left(u^2 + \theta^2\right) dx\right)^2 \left\| u \right\|_{L^2(0,l)}^2 + \frac{C_{\alpha_1}}{\varepsilon_7} \left(h_1 \circ u(t)\right) \\
\leq & \ \varepsilon_7 \left(\int_0^l \left(u^2 + \theta^2\right) dx\right)^2 \left\| u \right\|_{L^2(0,l)}^2 + \frac{C_p C_{\alpha_1}}{\varepsilon_7} \left(h_1 \circ u_{xx}(t)\right),
\end{aligned}$$

and with the same approach and (2.9), we have

$$\begin{aligned}
& \int_0^l \left(\left(\int_0^l \left(u^2 + \theta^2\right) dx\right) \theta(t)\right) \int_0^t g_2(t-s)\, \left(\theta(t) - \theta(s)\right) ds dx \qquad (2.20) \\
\leq & \ \varepsilon_7 \left(\int_0^l \left(u^2 + \theta^2\right) dx\right)^2 \left\| \theta \right\|_{L^2(0,l)}^2 + \frac{C_p C_{\alpha_2}}{\varepsilon_7} \left(h_2 \circ \theta_x(t)\right),
\end{aligned}$$

where C_p is the Poincaré constant. From (2.19) and (2.20), we deduce

$$\begin{aligned}
& \int_0^l \left(\left(\int_0^l \left(u^2 + \theta^2\right) dx\right) u(t)\right) \int_0^t g_1(t-s)\, \left(u(t) - u(s)\right) ds dx \qquad (2.21) \\
& + \int_0^l \left(\left(\int_0^l \left(u^2 + \theta^2\right) dx\right) \theta(t)\right) \int_0^t g_2(t-s)\, \left(\theta(t) - \theta(s)\right) ds dx \\
\leq & \ \varepsilon_7 \left(\int_0^l \left(u^2 + \theta^2\right) dx\right)^3 + \frac{C_p C_{\alpha_1}}{\varepsilon_7} \left(h_1 \circ u_{xx}(t)\right) + \frac{C_p C_{\alpha_2}}{\varepsilon_7} \left(h_2 \circ \theta_x(t)\right),
\end{aligned}$$

also, by using (2.1) and (2.5), we have

$$\left(\int_0^l \left(u^2+\theta^2\right) dx\right)^2 + 2\left(\int_0^l u\theta dx\right)^2 \leq E\left(0\right), \tag{2.22}$$

using (2.22) in (2.21), we get

$$\begin{aligned} & \int_0^l \left(\left(\left(\int_0^l \left(u^2+\theta^2\right) dx\right) u\left(t\right)\right) \int_0^t g_1\left(t-s\right)\left(u\left(t\right)-u\left(s\right)\right) dsdx \\ & + \int_0^l \left(\left(\left(\int_0^l \left(u^2+\theta^2\right) dx\right) \theta\left(t\right)\right) \int_0^t g_2\left(t-s\right)\left(\theta\left(t\right)-\theta\left(s\right)\right) dsdx \\ \leq \; & \varepsilon_7\sqrt{E\left(0\right)}\left(\int_0^l \left(u^2+\theta^2\right) dx\right)^2 + \frac{C_pC_{\alpha_1}}{\varepsilon_7}\left(h_1\circ u_{xx}\left(t\right)\right) + \frac{C_pC_{\alpha_2}}{\varepsilon_7}\left(h_2\circ \theta_x\left(t\right)\right). \end{aligned} \tag{2.23}$$

Using Cauchy–Schwarz, Young and Poincaré's inequalities, we have, for any $\varepsilon_7 > 0$

$$\begin{aligned} & \int_0^l \left(2\left(\int_0^l u\theta dx\right) \theta\ \left(t\right)\right) \int_0^t g_1\left(t-s\right)\left(u\left(t\right)-u\left(s\right)\right) dsdx \\ \leq \; & \varepsilon_7\left(\int_0^l u\theta dx\right)^2 \left\|\theta\right\|^2_{L^2(0,l)} + \frac{C_pC_{\alpha_1}}{\varepsilon_7}\left(h_1\circ u_{xx}\left(t\right)\right), \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} & \int_0^l \left(\left(\left(\int_0^l u\theta dx\right) u\ \left(t\right)\right) \int_0^t 2g_2\left(t-s\right)\left(\theta\left(t\right)-\theta\left(s\right)\right) dsdx \\ \leq \; & \varepsilon_7\left(\int_0^l u\theta dx\right)^2 \left\|u\right\|^2_{L^2(0,l)} + \frac{C_pC_{\alpha_2}}{\varepsilon_7}\left(h_2\circ \theta_x\left(t\right)\right), \end{aligned} \tag{2.25}$$

then by adding (2.24) and (2.25) and using (2.22), we obtain

$$\begin{aligned} & \int_0^l \left(2\left(\int_0^l u\theta dx\right) \theta\ \left(t\right)\right) \int_0^t g_1\left(t-s\right)\left(u\left(t\right)-u\left(s\right)\right) dsdx \\ & + \int_0^l \left(2\left(\int_0^l u\theta dx\right) u\ \left(t\right)\right) \int_0^t g_2\left(t-s\right)\left(\theta\left(t\right)-\theta\left(s\right)\right) dsdx \\ \leq \; & \varepsilon_7\sqrt{E\left(0\right)}\left(\int_0^l u\theta dx\right)^2 + \frac{C_pC_{\alpha_1}}{\varepsilon_7}\left(h_1\circ u_{xx}\left(t\right)\right) + \frac{C_pC_{\alpha_2}}{\varepsilon_7}\left(h_2\circ \theta_x\left(t\right)\right). \end{aligned} \tag{2.26}$$

Using Cauchy–Schwarz, Young and Poincaré's inequalities and by using $h_1\left(t\right) = \left(\alpha_1 g_1\left(t\right) - g_1'\left(t\right)\right)$

and $0 < \alpha_1 < 1$, we have, for any $\varepsilon_9 > 0$

$$
\begin{aligned}
& -\int_0^l u_t(t) \int_0^t g_1'(t-s)\,(u(t)-u(s))\,dsdx \qquad (2.27)\\
= & -\int_0^l u_t(t) \int_0^t h_1(t-s)\,(u(t)-u(s))\,dsdx + \int_0^l u_t(t) \int_0^t \alpha_1 g_1(t-s)\,(u(t)-u(s))\,dsdx \\
\leq & \ \varepsilon_9 \|u_t\|^2_{L^2(0,l)} + \frac{1}{2\varepsilon_9}\int_0^l \left(\int_0^t h_1(t-s)\,(u(t)-u(s))\,ds\right)^2 dx \\
& + \frac{\alpha_1^2}{2\varepsilon_9}\int_0^l \left(\int_0^t g_1(t-s)\,(u(t)-u(s))\,ds\right)^2 dx \\
\leq & \ \varepsilon_9 \|u_t\|^2_{L^2(0,l)} + \frac{\int_0^\infty h_1(t-s)\,ds}{2\varepsilon_9} h_1 \circ u(t) + \frac{\alpha_1^2 C_{\alpha_1}}{2\varepsilon_9} h_1 \circ u(t) \\
\leq & \ \varepsilon_9 \|u_t\|^2_{L^2(0,l)} + \frac{C_p}{2\varepsilon_9}\left((1-l_1)+g_1(0)+C_{\alpha_1}\right) h_1 \circ u_{xx}(t),
\end{aligned}
$$

and also, we have

$$
-\int_0^l \theta_t(t) \int_0^t g_2'(t-s)\,(\theta(t)-\theta(s))\,dsdx \leq \varepsilon_9 \|\theta_t\|^2_{L^2(0,l)} + \frac{C_p}{2\varepsilon_9}\left((1-l_2)+g_2(0)+C_{\alpha_2}\right) h_2 \circ \theta_x(t). \qquad (2.28)
$$

Using (2.15), (2.16), (2.17), (2.18), (2.23), (2.26), (2.27), (2.28), in (2.25), then, we get the result.

□

Applying the same arguments as in [14], we can get the following lemma.

Lemma 2.6. *Assume (A1) and (A1) hold, the function $\chi(t)$ defined by*

$$
\chi(t) = \int_0^l \int_0^t \sigma_1(t-s)\,|u_{xx}(s)|^2\,dsdx + \int_0^l \int_0^t \sigma_2(t-s)\,|\theta_x(s)|^2\,dsdx,
$$

where $\sigma_1(t) = \int_t^\infty g_1(s)\,ds$ and $\sigma_2(t) = \int_t^\infty g_2(s)\,ds$, satisfies

$$
\chi'(t) \leq -\frac{1}{2} g_1 \circ u_{xx}(t) - \frac{1}{2} g_2 \circ \theta_x(t) + 3(1-l_1)\,\|u_{xx}(t)\|^2_{L^2(0,l)} + 3(1-l_2)\,\|\theta_x(t)\|^2_{L^2(0,l)}. \qquad (2.29)
$$

To complete the proof of our main result, we will introduce the following Lyapunov functional $\mathcal{L}$

$$
\mathcal{L}(t) = N\,E(t) + N_1 I_1(t) + N_2 I_2(t), \qquad (2.30)
$$

where N, N_1 and N_2 are positive constants to be fixed later.

Lemma 2.7. *For N large enough, there exist two positive constants α_1 and α_2 such that*

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t). \tag{2.31}$$

Proof. Let's define the following functional

$$\mathcal{L}_1(t) = N_1 I_1(t) + N_2 I_2(t).$$

Using Cauchy–Schwarz,Young's and Poincaré's inequalities, we obtain

$$\begin{aligned}
|\mathcal{L}_1(t)| \leq\ & \frac{N_1}{2}\|u\|^2_{L^2(0,l)} + \frac{1}{2}[N_1+N_2]\|u_t\|^2_{L^2(0,l)} \\
&+ \frac{N_1}{2}\|\theta\|^2_{L^2(0,l)} + \frac{1}{2}[N_1+N_2]\|\theta_t\|^2_{L^2(0,l)} \\
&+ \frac{N_2 C_p}{2}(1-l_1)\, g_1 \circ u_{xx}(t) + \frac{N_2 C_p}{2}(1-l_2)\, g_2 \circ \theta_x(t).
\end{aligned}$$

Then by (2.1) and the properties of the functions g_1 and g_2, we get

$$|\mathcal{L}_1(t)| \leq c\, E(t).$$

Consequently

$$|\mathcal{L}(t) - N\, E(t)| \leq c\, E(t),$$

which implies that

$$(N-c)\, E(t) \leq \mathcal{L}(t) \leq (N+c)\, E(t).$$

Choosing N large enough, then we have (2.31). □

Proof of Theorem 2.2: Let $g_{0,1} = \int_0^t g_1(s)\, ds$ and $g_{0,2} = \int_0^t g_2(s)\, ds$, differentiating (2.30), exploiting (2.6) and (2.12), and applying Poincaré's inequality and by using $g_i' = \alpha_i g_i - h_i$; $i \in \{1,2\}$, we obtain the following estimates:

$$\begin{aligned}
\mathcal{L}'(t) \leq\ & -\frac{N}{2} g_1(t)\|u_{xx}\|^2_{L^2(0,l)} - \frac{N}{2} g_2(t)\|\theta_x\|^2_{L^2(0,l)} + \frac{N\alpha_1}{2} g_1 \circ u_{xx}(t) + \frac{N\alpha_2}{2} g_2 \circ \theta_x(t) \\
& - \left(\begin{array}{c} \frac{N}{2} - \left[\frac{N_1}{\varepsilon_1} + \left(1 + \frac{2C_p}{\varepsilon_7} + \frac{[1-l_1]^2}{\varepsilon_5} + \frac{1}{\varepsilon_3} + \frac{C_p}{2\varepsilon_9}\right) N_2\right] C_{\alpha_1} \\ -\dfrac{N_2 C_p}{2\varepsilon_9}\left((1-l_1) + g_1(0)\right) \end{array} \right) (h_1 \circ u_{xx}(t)) \\
& - \left(\begin{array}{c} \frac{N}{2} - \left[\frac{N_1}{\varepsilon_1} + \left(1 + \frac{2C_p}{\varepsilon_7} + \frac{[1-l_2]^2}{\varepsilon_5} + \frac{1}{\varepsilon_3} + \frac{C_p}{2\varepsilon_9}\right) N_2\right] C_{\alpha_2} \\ -\dfrac{N_2 C_p}{2\varepsilon_9}\left((1-l_2) + g_2(0)\right) \end{array} \right) (h_2 \circ \theta_x(t)) \\
& - \left((g_{0,1} - \varepsilon_9) N_2 - N_1\right)\|u_t\|^2_{L^2(0,l)} - \left((g_{0,2} - \varepsilon_9) N_2 - N_1\right)\|\theta_t\|^2_{L^2(0,l)} \\
& - \left((1-\varepsilon_1) N_1 - (\varepsilon_3 + \varepsilon_5) N_2\right)\|u_{xx}\|^2_{L^2(0,l)} - \left((1-\varepsilon_1) N_1 - (\varepsilon_3 + \varepsilon_5) N_2\right)\|\theta_x\|^2_{L^2(0,l)} \\
& - \left(N_1 - \varepsilon_7 N_2 \sqrt{E(0)}\right)\left(\int_0^l (u^2 + \theta^2)\, dx\right)^2 - \left(4N_1 - \varepsilon_7 N_2 \sqrt{E(0)}\right)\left(\int_0^l u\theta\, dx\right)^2.
\end{aligned}$$

Now, we will choose carefully the constants, first we choose

$$\varepsilon_3 = \varepsilon_5 = \frac{\varepsilon_1}{2},\ \varepsilon_1 = \frac{N_1}{2\left(N_1 + N_2\right)} \quad and\ \varepsilon_7 = \frac{N_1}{2N_2\sqrt{E\left(0\right)}}.$$

Clearly $0 < \dfrac{\alpha_i g_i^2\left(s\right)}{\alpha_i g_i\left(s\right) - g_i'\left(s\right)} < \dfrac{\alpha_i g_i^2\left(s\right)}{-g_i'\left(s\right)}$; $i \in \{1, 2\}$. Then for every $s \in [0, \infty)$

$$\lim_{\alpha_i \to 0} \frac{\alpha_i g_i^2\left(s\right)}{\alpha_i g_i\left(s\right) - g_i'\left(s\right)} = 0,$$

which, noting $\dfrac{\alpha_i g_i^2\left(s\right)}{\alpha_i g_i\left(s\right) - g_i'\left(s\right)} < g_i\left(s\right)$ and using the Lebesque dominated convergence theorem, gives us

$$\lim_{\alpha_i \to 0} \alpha_i C_{\alpha_i} = \lim_{\alpha_i \to 0} \int_0^\infty \frac{\alpha_i g_i^2\left(s\right)}{\alpha_i g_i\left(s\right) - g_i'\left(s\right)} ds = 0.$$

Hence there exists some α_0 $(0 < \alpha_0 < 1)$ such that if $\alpha_i < \alpha_0$ then

$$\alpha_i C_{\alpha_i} < \frac{1}{8\left[\frac{N_1}{\varepsilon_1} + \left(1 + \frac{2C_p}{\varepsilon_7} + \frac{1}{\varepsilon_5} + \frac{1}{\varepsilon_3} + \frac{C_p}{2\varepsilon_9}\right) N_2\right]}.$$

At last, we take N large enough and choose N_1, N_2, ε_9, α_1 and α_2 satisfying

$$\begin{array}{l} \frac{1}{4}N - \frac{N_2 C_p}{2\varepsilon_9}\left(1 + \max(l_1, l_2) + \max(g_1\left(0\right), g_2\left(0\right))\right) > 0,\ \alpha_1 = \alpha_2 = \frac{1}{2N} < \alpha_0 \\ N_1 = 10\left(1 - l_3\right),\ N_2 = \frac{2N_1}{g_0},\ \varepsilon_9 = \frac{1}{4} g_0\ where\ \left\{ \begin{array}{l} g_0 = \min\left(g_{0,1}, g_{0,2}\right) \\ l_3 = \min\left(l_1, l_2\right) \end{array} \right., \\ \left(\left(g_{0,1} - \varepsilon_9\right) N_2 - N_1\right) > \frac{1}{2}N_1 \quad and \quad \left(\left(g_{0,2} - \varepsilon_9\right) N_2 - N_1\right) > \frac{1}{2}N_1. \end{array}$$

Consequently, using inequality and (2.1), we get, $\forall t \geq t_0$

$$\begin{aligned} \mathcal{L}'\left(t\right) \leq\ & -5\left(1 - l_3\right)\left(\int_0^l \left(u^2 + \theta^2\right) dx\right)^2 - 5\left(1 - l_3\right)\left(\int_0^l u\theta dx\right)^2 \\ & -5\left(1 - l_3\right)\left\|u_{xx}\right\|_{L^2(0,l)}^2 - 5\left(1 - l_3\right)\left\|\theta_x\right\|_{L^2(0,l)}^2 - 5\left(1 - l_3\right)\left\|u_t\right\|_{L^2(0,l)}^2 \\ & -5\left(1 - l_3\right)\left\|\theta_t\right\|_{L^2(0,l)}^2 + \frac{1}{4} g_1 \circ u_{xx}\left(t\right) + \frac{1}{4} g_2 \circ \theta_x\left(t\right), \end{aligned} \tag{2.32}$$

using (2.5) and (2.4) to conclude that for any $t \geq t_0$

$$\begin{aligned} & \int_0^t g_1\left(s\right) \int_0^l \left|u_{xx}\left(t\right) - u_{xx}\left(t - s\right)\right|^2 dxds + \int_0^t g_2\left(s\right) \int_0^l \left|\theta_x\left(t\right) - \theta_x\left(t - s\right)\right|^2 dxds \\ \leq\ & -\frac{g_1\left(0\right)}{a_1} \int_0^t g_1'\left(s\right) \int_0^l \left|u_{xx}\left(t\right) - u_{xx}\left(t - s\right)\right|^2 dxds - \frac{g_2\left(0\right)}{a_2} \int_0^t g_2'\left(s\right) \int_0^l \left|\theta_x\left(t\right) - \theta_x\left(t - s\right)\right|^2 dxds \\ \leq\ & -cE'\left(t\right), \end{aligned}$$

which can be used in (2.32) and then take $F(t) = \mathcal{L}(t) + c\,E(t)$, which is clearly equivalent to $E(t)$, to get, for some constant $m > 0$ and for all $t \geq t_0$

$$\begin{aligned} \mathcal{L}'(t) &\leq -m\,E(t) + c\,g_1 \circ u_{xx}(t) + c\,g_2 \circ \theta_x(t) \\ &\leq -m\,E(t) - cE'(t) + c\int_{t_0}^{t} g_1(s)\,\|u_{xx}(t) - u_{xx}(t-s)\|^2\,ds \\ &\quad + c\int_{t_0}^{t} g_2(s)\,\|\theta_x(t) - \theta_x(t-s)\|^2\,ds. \end{aligned}$$

Then

$$\begin{aligned} F'(t) &\leq -m\,E(t) + c\int_{t_0}^{t} g_1(s)\,\|u_{xx}(t) - u_{xx}(t-s)\|^2\,ds \\ &\quad + c\int_{t_0}^{t} g_2(s)\,\|\theta_x(t) - \theta_x(t-s)\|^2\,ds. \end{aligned} \tag{2.33}$$

At this stage, we consider two cases.

(**I**) $H(t)$ **is linear**:

By multiplying (2.33) by $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$, and using $(A2)$ and (2.5), we obtain

$$\begin{aligned} \xi(t)F'(t) &\leq -m\xi(t)E(t) + c\xi(t)\int_{t_0}^{t} g_1(s)\,\|u_{xx}(t) - u_{xx}(t-s)\|^2\,ds \\ &\quad + c\xi(t)\int_{t_0}^{t} g_2(s)\,\|\theta_x(t) - \theta_x(t-s)\|^2\,ds; \\ &\leq -m\xi(t)E(t) + c\int_{t_0}^{t} \xi_1(s)g_1(s)\,\|u_{xx}(t) - u_{xx}(t-s)\|^2\,ds \\ &\quad + c\int_{t_0}^{t} \xi_2(s)g_2(s)\,\|\theta_x(t) - \theta_x(t-s)\|^2\,ds; \\ &\leq -m\xi(t)E(t) - c\int_{t_0}^{t} g_1'(s)\,\|u_{xx}(t) - u_{xx}(t-s)\|^2\,ds \\ &\quad - c\int_{t_0}^{t} g_2'(s)\,\|\theta_x(t) - \theta_x(t-s)\|^2\,ds; \\ &\leq -m\xi(t)E(t) - cE'(t), \end{aligned}$$

which gives, as $\xi(t)$ is positive and nonincreasing

$$\frac{d}{dt}\left(\xi(t)F(t) + c\,E(t)\right) \leq -m\xi(t)E(t); \qquad \forall t \geq t_0.$$

Hence, using the fact that $\xi F + c\,E \sim E$, we easily obtain

$$E(t) \leq C_0\,e^{-C_1\int_{t_0}^{t}\xi(s)\,ds};\ \forall t \geq t_0. \tag{2.34}$$

Finally we get (2.2).

(**II**) $H(t)$ **is nonlinear** : First, by using (2.29), (2.8) and (2.9) we deduce that

$$L(t) = \mathcal{L}(t) + \chi(t),$$

is nonnegative and satisfies, for all $t \geq t_0$

$$\begin{aligned} L'(t) &\leq -(1-l_1)\|u_{xx}\|^2_{L^2(0,l)} - (1-l_2)\|\theta_x\|^2_{L^2(0,l)} - 5(1-l_3)\left(\int_0^l \left(u^2+\theta^2\right)dx\right)^2 \\ &\quad -5(1-l_3)\left(\int_0^l u\theta dx\right)^2 - 5(1-l_3)\|u_t\|^2_{L^2(0,l)} - 5(1-l_3)\|\theta_t\|^2_{L^2(0,l)} \\ &\quad -\frac{1}{4}g_1 \circ u_{xx}(t) - \frac{1}{4}g_2 \circ \theta_x(t); \\ &\leq -b_3 E(t), \end{aligned}$$

where b_3 is some positive constant. Therefore

$$b_3 \int_{t_0}^t E(s)\,ds \leq L(t_0) - L(t) \leq L(t_0),$$

this implies that

$$\int_0^\infty E(s)\,ds < \infty. \tag{2.35}$$

Define $K_1(t)$ and $K_2(t)$ by

$$\begin{aligned} K_1(t) &= q\int_{t_0}^t \int_0^l |u_{xx}(t) - u_{xx}(t-s)|^2 dxds, \\ K_2(t) &= q\int_{t_0}^t \int_0^l |\theta_x(t) - \theta_x(t-s)|^2 dxds, \end{aligned}$$

where (2.35) allows for a constant $0 < q < 1$ chosen so that, for all $t \geq t_0$

$$K_1(t) < 1 \quad \text{and} \quad K_2(t) < 1. \tag{2.36}$$

Also, we define $\lambda_1(t)$ and $\lambda_2(t)$ by

$$\begin{aligned} \lambda_1(t) &= -\int_{t_0}^t g_1'(s)\int_0^l |u_{xx}(t) - u_{xx}(t-s)|^2 dxds, \\ \lambda_2(t) &= -\int_{t_0}^t g_2'(s)\int_0^l |\theta_x(t) - \theta_x(t-s)|^2 dxds. \end{aligned}$$

It is easy to verify that $\lambda_1(t) \leq -c\,E'(t)$ and $\lambda_2(t) \leq -c\,E'(t)$. Noting that $H_i(t)$; $i \in \{1,2\}$; is strictly convex on $(0,r]$ and $H_i(t) = 0$, we can infer that

$$H_i(\upsilon x) \leq \upsilon H_i(x),$$

provided $0 < \upsilon < 1$ and $x \in (0,r]$. We denote an extension of H_i by $\overline{H_i}$; $i \in \{1,2\}$; such that $\overline{H_i}$ is strictly increasing and strictly convex C^2 function on $(0,\infty)$. By using Assumption

$(A2)$, (2.36) and Jensen's inequality, we can obtain, in the case where $K_1(t), K_2(t) > 0$ for all $t \geq t_0$

$$\begin{aligned}
\lambda_1(t) &= \frac{1}{qK_1(t)} \int_{t_0}^{t} K_1(t) \left[-g_1'(s)\right] \int_0^l q \left|u_{xx}(t) - u_{xx}(t-s)\right|^2 dxds \\
&\geq \frac{1}{qK_1(t)} \int_{t_0}^{t} K_1(t) \, \xi_1(s) \, H_1(g_1(s)) \int_0^l q \left|u_{xx}(t) - u_{xx}(t-s)\right|^2 dxds \\
&\geq \frac{\xi_1(t)}{qK_1(t)} \int_{t_0}^{t} H_1(K_1(t) \; g_1(s)) \int_0^l q \left|u_{xx}(t) - u_{xx}(t-s)\right|^2 dxds \\
&\geq \frac{\xi_1(t)}{q} H_1\left(q \int_{t_0}^{t} g_1(s) \int_0^l \left|u_{xx}(t) - u_{xx}(t-s)\right|^2 dxds \right) \\
&= \frac{\xi_1(t)}{q} \overline{H_1}\left(q \int_{t_0}^{t} g_1(s) \int_0^l \left|u_{xx}(t) - u_{xx}(t-s)\right|^2 dxds \right),
\end{aligned}$$

this implies that

$$\int_{t_0}^{t} g_1(s) \int_0^l \left|u_{xx}(t) - u_{xx}(t-s)\right|^2 dxds \leq \frac{1}{q} \overline{H_1}^{-1} \left(\frac{q\lambda_1(t)}{\xi_1(t)} \right), \tag{2.37}$$

and with the same approach, we have

$$\int_{t_0}^{t} g_2(s) \int_0^l \left|\theta_x(t) - \theta_x(t-s)\right|^2 dxds \leq \frac{1}{q} \overline{H_2}^{-1} \left(\frac{q\lambda_2(t)}{\xi_2(t)} \right). \tag{2.38}$$

We notice that (2.37) is verified $if \; K_1(t) = 0$ and that (2.38) is verified $if \; K_2(t) = 0$. Then (2.33) becomes

$$F'(t) \preceq -m \, E(t) + \frac{1}{q} \overline{H_1}^{-1} \left(\frac{q\lambda_1(t)}{\xi_1(t)} \right) + \frac{1}{q} \overline{H_2}^{-1} \left(\frac{q\lambda_2(t)}{\xi_2(t)} \right); \qquad \forall t \geq t_0. \tag{2.39}$$

Denote

$$H_0(t) = \min\{\overline{H}_1', \overline{H}_2'\}.$$

For $\varepsilon_0 < r$, using (2.39), and the fact that $E'(t) \leq 0$, $\overline{H_i}' > 0$, $\overline{H_i}'' > 0$, $i \in \{1, 2\}$; we find that the functional F_1 defined by

$$F_1(t) = \overline{H_0}\left(\varepsilon_0 \frac{E(t)}{E(0)} \right) F(t) + E(t),$$

is equivalent to E and

$$\begin{aligned}
F_1'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)} H_0'\left(\varepsilon_0 \frac{E(t)}{E(0)} \right) F(t) + H_0\left(\varepsilon_0 \frac{E(t)}{E(0)} \right) F'(t) + E'(t) \\
&\leq -mE(t) H_0\left(\varepsilon_0 \frac{E(t)}{E(0)} \right) + cH_0\left(\varepsilon_0 \frac{E(t)}{E(0)} \right) \overline{H}_1^{-1} \left(\frac{q\lambda_1(t)}{\xi_1(t)} \right) \\
&\quad + cH_0\left(\varepsilon_0 \frac{E(t)}{E(0)} \right) \overline{H}_2^{-1} \left(\frac{q\lambda_2(t)}{\xi_2(t)} \right).
\end{aligned} \tag{2.40}$$

Let $\overline{H_i}^*$ be the convex conjugate of $\overline{H_i}$, $i \in \{1,2\}$ in the sense of Young (see [[7] , pp.61-64]) then

$$\overline{H_i}^*(s) = s\left(\overline{H_i'}\right)^{-1}(s) - \overline{H_i}\left[\left(\overline{H_i'}\right)^{-1}(s)\right], \tag{2.41}$$

and $\overline{H_i}^*$ satisfied the following Young's inequality

$$AB_i \leq \overline{H_i}^*(A_i) + \overline{H_i}(B_i). \tag{2.42}$$

With $A = H_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and $B_i = \overline{H_1}^{-1}\left(\dfrac{q\lambda_i(t)}{\xi_i(t)}\right)$, using (2.40), (2.41) and (2.42), we arrive at

$$\begin{aligned}
F_1'(t) &\leq -mE(t)H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + c\overline{H}_1^*\left(H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right) + c\frac{q\lambda_1(t)}{\xi_1(t)} \\
&\quad + c\overline{H}_2^*\left(H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right) + c\frac{q\lambda_2(t)}{\xi_2(t)} \\
&\leq -mE(t)H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + cH_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)(\overline{H}_1')^{-1}\left(H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right) \\
&\quad + c\frac{q\lambda_1(t)}{\xi_1(t)} + cH_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)(\overline{H}_2')^{-1}\left(H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right) + c\frac{q\lambda_2(t)}{\xi_2(t)} \\
&\leq -mE(t)H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + cH_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)(\overline{H}_1')^{-1}\left(\overline{H}_1'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right) \\
&\quad + c\frac{q\lambda_1(t)}{\xi_1(t)} + cH_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)(\overline{H}_2')^{-1}\left(\overline{H}_2'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right) + c\frac{q\lambda_2(t)}{\xi_2(t)} \\
&\leq -(mE(0) - c\varepsilon_0)\frac{E(t)}{E(0)}H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + cq\left(\frac{\lambda_1(t)}{\xi_1(t)} + \frac{\lambda_2(t)}{\xi_2(t)}\right).
\end{aligned} \tag{2.43}$$

Then, we multiply (2.43) by $\xi(t) = \min\{\xi_1, \xi_2(t)\}$ to get

$$\begin{aligned}
\xi(t)F_1'(t) &\leq -(mE(0) - c\varepsilon_0)\xi(t)\frac{E(t)}{E(0)}H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + cq\left(\lambda_1(t) + \lambda_2(t)\right) \\
&\leq -(mE(0) - c\varepsilon_0)\xi(t)\frac{E(t)}{E(0)}H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) - cE'(t).
\end{aligned}$$

Consequently, with $F_2 = \xi F_1 + cE$, which satisfies, for some α_1, $\alpha_2 > 0$

$$\alpha_1 F_2(t) \leq E(t) \leq \alpha_2 F_2(t) \tag{2.44}$$

and with a suitable choice of ε_0, we obtain, for some constant $k > 0$ and for all $t \geq t_0$,

$$F_2'(t) \leq -k\xi(t)\frac{E(t)}{E(0)}H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) := -k\xi(t)H_3\left(\frac{E(t)}{E(0)}\right) \tag{2.45}$$

where

$$H_3(t) = tH_0(\varepsilon_0 t).$$

We know that, if $0 \leq \varepsilon_0 \frac{E(t)}{E(0)} < r$, for any $t > 0$

$$\begin{aligned} H_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) &= \min\left\{\overline{H}_1'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right), \overline{H}_2'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right\} \\ &= \min\left\{H_1'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right), H_2'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right\}. \end{aligned}$$

Since $H_3'(t) = H_0(\varepsilon_0 t) + \varepsilon_0 t H_0'(\varepsilon_0 t)$, then, using the strict convexity of H_0 on $(0, r]$, we know that $H_3'(t), H_3(t) > 0$ on $(0, 1]$. Then

$$R(t) = \frac{\alpha_1 F_2(t)}{E(0)},$$

taking in account (2.44) and (2.45), satisfies

$$R(t) \sim E(t), \tag{2.46}$$

and, for some $k_1 > 0$,

$$R'(t) \leq -k_1 \xi(t)\; H_3(R(t)), \qquad \forall t \geq t_0.$$

Then, the integration over (t_0, t) yields

$$\int_0^t \frac{-R'(s)}{H_3(R(s))} ds \geq k_1 \int_0^t \xi(s) ds \Rightarrow \int_{\varepsilon_0 R(t)}^{\varepsilon_0 R(0)} \frac{1}{s H_0(s)} ds \geq k_1 \int_0^t \xi(s) ds.$$

$$\Longrightarrow R(t) \preceq \frac{1}{\varepsilon_0} H_4^{-1}\left(k_1 \int_{t_0}^t \xi(s)\, ds\right), \tag{2.47}$$

where $H_4(t) = \displaystyle\int_t^r \frac{1}{s\, H_0(s)} ds$. Here, we have used, based on the properties of H_1 and H_2, $\varepsilon_0 \frac{E(t)}{E(0)} < r$, 2.44, the fact that H_4 is strictly decreasing function on $(0, r]$ and $\lim\limits_{t \to 0} H_4(t) = +\infty$. By using (2.46) and (2.47) , estimate (2.2) is established. □

3 Conclusion and open question

In this paper we established a general decay for the nonlocal nonlinear suspension bridges model in one dimensional space under two memories, our method of proof is mainly based on multipliers techniques. It will be interesting to check the decay rates for both components of the solutions if one of the memory terms $g_i(t)$ is completely zero.

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