Uniform decay rates of a Bresse thermoelastic system in the whole space

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Abstract

In this paper, we investigate the decay properties of the thermoelastic Bresse system in the whole space. We consider many cases depending on the parameters of the model and we establish new decay rates. We need to mention here that, in some cases we don't have the regularity-loss phenomena as in the previous works in the literature. To prove our results, we use the energy method in the Fourier space to build a very delicate Lyapunov functionals that give the desired results.

Keywords: Bresse system, energy method, Lyapunov functional.
AMS Subject Classifications: 93D20, 35B40.

1 Introduction

In this paper, we consider the following initial value of the thermoelastic Bresse system. Namely, our concern is the asymptotic behavior of the solution of the following:

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi - l\,w)_x - k_0^2 l\,(w_x - l\varphi) = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi - l\,w) + m\theta_x = 0, \\ w_{tt} - k_0^2\,(w_x - l\varphi)_x - l\,(\varphi_x - \psi - l\,w) + \gamma\, w_t = 0, \\ \theta_t - k_1\theta_{xx} + m\psi_{t\,x} = 0, \end{cases} \tag{1.1}$$

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with the initial data

$$(\varphi, \varphi_t, \psi, \psi_t, w, w_t, \theta)(x, 0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \theta_0), \tag{1.2}$$

where $(x;t) \in \mathbb{R} \times \mathbb{R}^+$, the functions φ, ψ and w, denote respectively the vertical displacement of the beam, the rotation angle and the longitudinal displacement, the function θ is the temperature difference and a, l, m, k_0, k_1 and γ are positive constants.

We consider many cases depending on the parameters of the model, and we prove that the solution $U = (\varphi, \varphi_t, \psi, \psi_t, w, w_t, \theta)^T$ is decaying as follow:

$$\left\|\partial_x^k U(t)\right\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|U_0\|_{L^1} + Ce^{-ct}\left\|\partial_x^k U_0\right\|_{L^2}, \; if \;\; (3.1) \; is \; satisfied, \tag{1.3}$$

$$\left\|\partial_x^k U(t)\right\|_{L^2} \leq C(1+t)^{-\frac{1}{8}-\frac{k}{4}} \|U_0\|_{L^1} + Ce^{-ct}\left\|\partial_x^k U_0\right\|_{L^2}, \; if \; (3.2) \; or \; (3.3) \; is \; satisfied, \tag{1.4}$$

$$\left\|\partial_x^k U(t)\right\|_{L^2} \leq C(1+t)^{-\frac{1}{8}-\frac{k}{4}} \|U_0\|_{L^1} + C(1+t)^{-\frac{\delta}{2}}\left\|\partial_x^{k+\delta} U_0\right\|_{L^2}, \; if \; (3.4) \; or \; (3.5) \; is \; satisfied, \tag{1.5}$$

$$\left\|\partial_x^k U(t)\right\|_{L^2} \leq C(1+t)^{-\frac{1}{12}-\frac{k}{6}} \|U_0\|_{L^1} + C(1+t)^{-\frac{\delta}{2}}\left\|\partial_x^{k+\delta} U_0\right\|_{L^2}, \; if \; (3.6) \; or \; (3.7) \; is \; satisfied. \tag{1.6}$$

where k and δ are nonnegative integers, C and c are two positive constants and $U_0 = U(x;0)$.

In order to prove the above estimates (1.3)- (1.6), we use a Fourier energy method as well as a suitable linear combination of series of energy estimates, to show that the solution in the Fourier image $\widehat{U}(\xi, t)$ satisfies the following estimates:

$$\left|\widehat{U}(\xi, t)\right|^2 \leq \begin{cases} C\, e^{-c\lambda_1(\xi)\, t}\left|\widehat{U}(\xi, 0)\right|^2; & if \; (3.1) \; is \; satisfied, \\ C\, e^{-c\lambda_2(\xi)\, t}\left|\widehat{U}(\xi, 0)\right|^2; & if \; (3.2) \;\; or \; (3.3) \; is \; satisfied, \\ C\, e^{-c\lambda_3(\xi)\, t}\left|\widehat{U}(\xi, 0)\right|^2; & if \; (3.4) \;\; or \; (3.5) \; is \; satisfied, \\ C\, e^{-c\lambda_4(\xi)\, t}\left|\widehat{U}(\xi, 0)\right|^2; & if \; (3.6) \;\; or \; (3.7) \; is \; satisfied, \end{cases}$$

where the functions $\lambda_i(\xi)$ $(i = 1..4)$ are defined in (3.2). It is well known in the literature, that the behavior of $\lambda_i(\xi)$ in the low frequencies determines the rate of decay of the solution, while its behavior for high frequencies gives the regularity restriction on the initial data see for instance ([1], [5], [11], [12]) and the references therein.

We need to mention here that the system has been considered by [11] where they showed that the solution decays as follows:

$$\left\|\partial_x^k U(t)\right\|_{L^2} \leq C(1+t)^{-\frac{1}{12}-\frac{k}{6}} \|U_0\|_{L^1} + C(1+t)^{-\frac{\delta}{2}}\left\|\partial_x^{k+\delta} U_0\right\|_{L^2} \quad if \;\; a = 1,$$

$$\left\|\partial_x^k U(t)\right\|_{L^2} \leq C(1+t)^{-\frac{1}{12}-\frac{k}{6}} \|U_0\|_{L^1} + C(1+t)^{-\frac{\delta}{4}}\left\|\partial_x^{k+\delta} U_0\right\|_{L^2} \quad if \;\; a \neq 1$$

Clearly we can see that our estimates improves the decay rates in [11]. One can see that, in some cases we don't have the regularity loss phenomena as in the previous works existing in the literature.

Our paper is organized as follow: In Section 2, we state the problem. The proofs are given in Section 3.

2 Statement of the Problem

In this section, as in [11], for simplicity, we write the system (1.1) as a first order (in time) system by introducing the following variables:

$$v = \varphi_x - \psi - l\,w; \quad u = \varphi_t; \quad z = a\,\psi_x; \quad y = \psi_t; \quad \phi = k_0\,(w_x - l\varphi)\,; \quad \eta = w_t.$$

Consequently, the system (1.1) can be rewritten into the following first order system

$$\begin{cases} v_t - u_x + y + l\,\eta = 0, \\ u_t - v_x - l\,k_0\,\phi = 0, \\ z_t - a\,y_x = 0, \\ y_t - a\,z_x - v + m\,\theta_x = 0, \\ \phi_t - \ k_0\,\eta_x + l\,k_0 u = 0, \\ \eta_t - k_0\,\phi_x - l\,v + \gamma\eta = 0, \\ \theta_t - k_1\theta_{xx} + m\,y_x = 0. \end{cases} \tag{2.1}$$

Taking the Fourier transform of (2.1), we find:

$$\begin{cases} \widehat{v}_t - i\xi\,\widehat{u} + \widehat{y} + l\,\widehat{\eta} = 0, \\ \widehat{u}_t - i\xi\,\widehat{v} - l\,k_0\,\widehat{\phi} = 0, \\ \widehat{z}_t - a\ i\xi\,\widehat{y} = 0, \\ \widehat{y}_t - a\,i\,\xi\,\widehat{z} - \widehat{v} + m\,i\xi\,\widehat{\theta} = 0, \\ \widehat{\phi}_t - \ k_0\,i\,\xi\,\widehat{\eta} + l\,k_0\widehat{u} = 0, \\ \widehat{\eta}_t - k_0\,i\xi\,\widehat{\phi} - l\,\widehat{v} + \gamma\,\widehat{\eta} = 0, \\ \widehat{\theta}_t + k_1\xi^2\widehat{\theta} + m\,i\,\xi\,\widehat{y} = 0. \end{cases} \tag{2.2}$$

Let us now define the following energy functional

$$\widehat{E}\,(\xi,t) = \frac{1}{2}\left(|\widehat{v}|^2 + |\widehat{u}|^2 + |\widehat{z}|^2 + |\widehat{y}|^2 + \left|\widehat{\phi}\right|^2 + |\widehat{\eta}|^2 + \left|\widehat{\theta}\right|^2\right)(\xi,t)\,, \tag{2.3}$$

We use the same procedure as in [11] to obtain:

$$\frac{d\widehat{E}\,(\xi,t)}{dt} = -k_1\xi^2\left|\widehat{\theta}\right|^2 - \gamma\,|\widehat{\eta}|^2\,. \tag{2.4}$$

3 Energy method and decay estimates

In this section, we show that the decay rate of the solution will depend on the wave speeds of the first two equations in the system (1.1) as well as on the coefficients l and k_0. For this reason, we will discuss seven cases:

$$\begin{aligned}
Case\ 1.\ a &= 1\, and\ l\, k_0 = 1, && (3.1)\\
Case\ 2.\ a &= 1\, and\ l\, k_0 < 1, && (3.2)\\
Case\ 3.\ a &= 1\, and\ l\, k_0 > 1\ and\ \left(1 + l^2\left(1 - k_0^2\right)\right) > 0, && (3.3)\\
Case\ 4.\ a &\neq 1\, and\ l\, k_0 \leq 1, && (3.4)\\
Case\ 5.\ a &\neq 1\, and\ l\, k_0 > 1\ and\ \left(1 + l^2\left(1 - k_0^2\right)\right) > 0, && (3.5)\\
Case\ 6.\ a &= 1\, and\ l\, k_0 > 1\ and\ \left(1 + l^2\left(1 - k_0^2\right)\right) \leq 0, && (3.6)\\
Case\ 7.\ a &\neq 1\, and\ l\, k_0 > 1\ and\ \left(1 + l^2\left(1 - k_0^2\right)\right) \leq 0. && (3.7)
\end{aligned}$$

In each case we use a delicate energy method to build the appropriate Lyapunov functionals in the Fourier space.

3.1 The energy method in the Fourier space

In this subsection, we give the pointwise estimates of the functional $\widehat{E}(\xi, t)$.We show an estimate for the Fourier image of the solution. This estimate will play the key role in proving our main result. The results are stated in the following proposition:

Proposition 3.1. *For any $t \geq 0$ and $\xi \in \mathbb{R}$, we have the following estimates*

$$\widehat{E}(\xi, t) \leq \begin{cases} C\, e^{-c\lambda_1(\xi)\, t}\widehat{E}(\xi, 0); & if\ (3.1)\ is\ satisfied, \\ C\, e^{-c\lambda_2(\xi)\, t}\widehat{E}(\xi, 0); & if\ (3.2)\ \ or\ (3.3)\ is\ satisfied, \\ C\, e^{-c\lambda_3(\xi)\, t}\widehat{E}(\xi, 0); & if\ (3.4)\ \ or\ (3.5)\ is\ satisfied, \\ C\, e^{-c\lambda_4(\xi)\, t}\widehat{E}(\xi, 0); & if\ (3.6)\ \ or\ (3.7)\ is\ satisfied, \end{cases} \tag{3.1}$$

where

$$\lambda_1(\xi) = \frac{\xi^2}{1+\xi^2}, \qquad \lambda_2(\xi) = \frac{\xi^4}{\left(1+\xi^2\right)^2}, \qquad \lambda_3(\xi) = \frac{\xi^4}{\left(1+\xi^2\right)^3}, \qquad \lambda_4(\xi) = \frac{\xi^6}{\left(1+\xi^2\right)^4}. \tag{3.2}$$

Here C and c are two positive constants.

The proof of proposition 3.1 will be given through several steps

Proof. Step1. Multiplying $(2.2)_6$ by $\imath\xi\,\widehat{\theta}$, we get

$$\begin{aligned}
0 &= \left\langle \widehat{\eta}_t, \imath\xi\,\widehat{\theta}\right\rangle - k_0\left\langle i\xi\,\widehat{\phi}, \imath\xi\,\widehat{\theta}\right\rangle - l\left\langle \widehat{v}, \imath\xi\,\widehat{\theta}\right\rangle + \gamma\left\langle \widehat{\eta}, \imath\xi\,\widehat{\theta}\right\rangle \\
&= \frac{\partial}{\partial t}\left\langle \widehat{\eta}, \imath\xi\,\widehat{\theta}\right\rangle + \left(\gamma + k_1\xi^2\right)\left\langle \widehat{\eta}, \imath\xi\,\widehat{\theta}\right\rangle - m\xi^2\left\langle \widehat{\eta}, \widehat{y}\right\rangle - k_0\,\xi^2\left\langle \widehat{\phi}, \widehat{\theta}\right\rangle - l\left\langle \widehat{v}, \imath\xi\,\widehat{\theta}\right\rangle,
\end{aligned}$$

then

$$l\left\langle \widehat{v}, \imath\xi\,\widehat{\theta}\right\rangle = \frac{\partial}{\partial t}\left\langle \widehat{\eta}, \imath\xi\,\widehat{\theta}\right\rangle + \left(\gamma + k_1\xi^2\right)\left\langle \widehat{\eta}, \imath\xi\,\widehat{\theta}\right\rangle - m\xi^2\left\langle \widehat{\eta}, \widehat{y}\right\rangle - k_0\,\xi^2\left\langle\, \widehat{\phi},\, \widehat{\theta}\right\rangle. \tag{3.3}$$

Multiplying $(2.2)_7$ by $\imath\xi\widehat{y}$, we have

$$\begin{aligned} 0 &= \left\langle \widehat{\theta}_t, \imath\xi\,\widehat{y}\right\rangle + k_1\xi^2\left\langle \widehat{\theta}, \imath\xi\widehat{y}\right\rangle + m\,\xi^2\,\left|\widehat{y}\right|^2 \qquad (3.4)\\ &= \frac{\partial}{\partial t}\left\langle \widehat{\theta}, \imath\xi\,\widehat{y}\right\rangle - \left\langle \widehat{\theta}, \imath\xi\,\left(a\,i\,\xi\,\widehat{z} + \widehat{v} - m\,i\xi\,\widehat{\theta}\right)\right\rangle + k_1\xi^2\left\langle \widehat{\theta}, \imath\xi\widehat{y}\right\rangle + m\,\xi^2\,\left|\widehat{y}\right|^2\\ &= \frac{\partial}{\partial t}\left\langle \widehat{\theta}, \imath\xi\,\widehat{y}\right\rangle + a\xi^2\left\langle \widehat{\theta}, \widehat{z}\right\rangle - \left\langle \widehat{\theta}, \imath\xi\widehat{v}\right\rangle - m\,\xi^2\,\left|\widehat{\theta}\right|^2 + k_1\xi^2\left\langle \widehat{\theta}, \imath\xi\widehat{y}\right\rangle + m\,\xi^2\,\left|\widehat{y}\right|^2, \end{aligned}$$

by using (3.3) and (3.4),

$$\begin{aligned} &\xi^2\,\left|\widehat{y}\right|^2 + \frac{1}{m}\frac{\partial}{\partial t}\operatorname{Re}\left\langle \widehat{\theta}, \imath\xi\,\widehat{y}\right\rangle + \frac{1}{ml}\frac{\partial}{\partial t}\left\langle \widehat{\eta}, \imath\xi\,\widehat{\theta}\right\rangle \qquad (3.5)\\ =\;& \xi^2\,\left|\widehat{\theta}\right|^2 - \frac{a\xi^2}{m}\operatorname{Re}\left\langle \widehat{\theta}, \widehat{z}\right\rangle - \frac{k_1\xi^2}{m}\operatorname{Re}\left\langle \widehat{\theta}, \imath\xi\,\widehat{y}\right\rangle\\ &- \frac{\left(\gamma + k_1\xi^2\right)}{ml}\operatorname{Re}\left\langle \widehat{\eta}, \imath\xi\,\widehat{\theta}\right\rangle + \frac{\xi^2}{l}\operatorname{Re}\left(\left\langle \widehat{\eta}, \widehat{y}\right\rangle\right) + \frac{k_0\,\xi^2}{ml}\operatorname{Re}\left\langle\, \widehat{\phi},\, \widehat{\theta}\right\rangle. \end{aligned}$$

For the case (3.1), we proceed as follow:

Multiplying (3.5) by $\frac{1}{(1+\xi^2)}$, applying Young's inequality and the following estimates, for any $\varepsilon_1 > 0$, given by

$$\frac{a\xi^2}{m\left(1+\xi^2\right)}\left|\operatorname{Re}\left\langle \widehat{\theta}, \widehat{z}\right\rangle\right| \le C\left(\varepsilon_1\right)\xi^2\left|\widehat{\theta}\right|^2 + \frac{\varepsilon_1}{2}\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{z}\right|^2,$$

$$\frac{k_1\xi^2}{m\left(1+\xi^2\right)}\left|\operatorname{Re}\left\langle \widehat{\theta}, \imath\xi\,\widehat{y}\right\rangle\right| = \frac{k_1}{m}\left|\operatorname{Re}\left\langle \xi\,\widehat{\theta}, \frac{\imath\xi^2}{\left(1+\xi^2\right)}\,\widehat{y}\right\rangle\right| \le C\,\xi^2\left|\widehat{\theta}\right|^2 + \frac{1}{4}\frac{\xi^2}{\left(1+\xi^2\right)}\,\left|\widehat{y}\right|^2,$$

$$\frac{\left(\gamma + k_1\xi^2\right)}{ml\left(1+\xi^2\right)}\left|\operatorname{Re}\left\langle \widehat{\eta}, \imath\xi\,\widehat{\theta}\right\rangle\right| \le C\,\xi^2\left|\widehat{\theta}\right|^2 + C\,\left|\widehat{\eta}\right|^2,$$

$$\frac{\xi^2}{l\left(1+\xi^2\right)}\left|\operatorname{Re}\left\langle \widehat{\eta}, \widehat{y}\right\rangle\right| \le C\,\left|\widehat{\eta}\right|^2 + \frac{1}{4}\frac{\xi^2}{\left(1+\xi^2\right)}\,\left|\widehat{y}\right|^2,$$

$$\frac{k_0\,\xi^2}{ml\left(1+\xi^2\right)}\left|\operatorname{Re}\left\langle\, \widehat{\phi},\, \widehat{\theta}\right\rangle\right| \le C\left(\varepsilon_1\right)\xi^2\left|\widehat{\theta}\right|^2 + \frac{\varepsilon_1}{2}\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{\phi}\right|^2,$$

then, we obtain

$$\frac{\xi^2}{\left(1+\xi^2\right)}\,\left|\widehat{y}\right|^2 + \frac{\partial}{\partial t}\mathcal{F}_1(\xi,t) \le C\left(\varepsilon_1\right)\xi^2\left|\widehat{\theta}\right|^2 + C\,\left|\widehat{\eta}\right|^2 + \varepsilon_1\frac{\xi^2}{\left(1+\xi^2\right)}\,\left|\widehat{z}\right|^2 + \varepsilon_1\frac{\xi^2}{\left(1+\xi^2\right)}\,\left|\widehat{\phi}\right|^2, \tag{3.6}$$

where
$$\mathcal{F}_1(\xi,t)=\frac{2}{m\left(1+\xi^2\right)}\operatorname{Re}\left\langle\widehat{\theta},\imath\xi\,\widehat{y}\right\rangle+\frac{2}{ml\left(1+\xi^2\right)}\left\langle\widehat{\eta},\imath\xi\,\widehat{\theta}\right\rangle. \tag{3.7}$$

For the cases (3.2- 3.5), then we proceed as follow:

Multiplying (3.5) by $\frac{\xi^2}{\left(1+\xi^2\right)^2}$, applying Young's inequality and the following estimates, for any $\varepsilon_1>0$, given by

$$\frac{a}{m}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle\widehat{\theta},\widehat{z}\right\rangle\right|\le C\left(\varepsilon_1\right)\xi^2\left|\widehat{\theta}\right|^2+\frac{\varepsilon_1}{2}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2,$$

$$\frac{k_1}{m}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle\imath\xi\,\widehat{\theta},\widehat{y}\right\rangle\right|\le C\,\xi^2\left|\widehat{\theta}\right|^2+\frac{1}{4}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2,$$

$$\frac{\left(\gamma+k_1\xi^2\right)\xi^2}{ml\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle\widehat{\eta},\imath\xi\,\widehat{\theta}\right\rangle\right|\le C\,\xi^2\left|\widehat{\theta}\right|^2+C\left|\widehat{\eta}\right|^2,$$

$$\frac{\xi^4}{l\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle\widehat{\eta},\widehat{y}\right\rangle\right|\le C\left|\widehat{\eta}\right|^2+\frac{1}{4}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2,$$

$$\frac{k_0\xi^4}{ml\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle\widehat{\phi},\widehat{\theta}\right\rangle\right|\le C\left(\varepsilon_1\right)\xi^2\left|\widehat{\theta}\right|^2+\frac{\varepsilon_1}{2}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2,$$

then, we obtain

$$\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2+\frac{\partial}{\partial t}\mathcal{F}_1(\xi,t)\le C\left(\varepsilon_1\right)\xi^2\left|\widehat{\theta}\right|^2+C\left|\widehat{\eta}\right|^2+\varepsilon_1\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2+\varepsilon_1\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2, \tag{3.8}$$

where
$$\mathcal{F}_1(\xi,t)=\frac{2\xi^2}{m\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle\widehat{\theta},\imath\xi\,\widehat{y}\right\rangle+\frac{2\xi^2}{ml\left(1+\xi^2\right)^2}\left\langle\widehat{\eta},\imath\xi\,\widehat{\theta}\right\rangle. \tag{3.9}$$

For the cases (3.6- 3.7), then we proceed as follow:

Multiplying (3.5) by $\frac{\xi^2}{\left(1+\xi^2\right)^2}$, applying Young's inequality and the following estimates, for any $\varepsilon_1>0$, given by

$$\frac{a\xi^4}{m\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle\widehat{\theta},\widehat{z}\right\rangle\right|\le C\left(\varepsilon_1\right)\xi^2\left|\widehat{\theta}\right|^2+\frac{\varepsilon_1}{2}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2,$$

$$\frac{k_1\xi^4}{m\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle\imath\xi\widehat{\theta},\,\widehat{y}\right\rangle\right|\le C\,\xi^2\left|\widehat{\theta}\right|^2+\frac{1}{4}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2,$$

$$\frac{\left(\gamma+k_1\xi^2\right)\xi^2}{ml\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle\widehat{\eta},\imath\xi\,\widehat{\theta}\right\rangle\right|\le C\,\xi^2\left|\widehat{\theta}\right|^2+C\left|\widehat{\eta}\right|^2,$$

$$\frac{\xi^4}{l\left(1+\xi^2\right)^2}\left|\mathrm{Re}\left\langle \widehat{\eta},\widehat{y}\right\rangle\right| \leq C\left|\widehat{\eta}\right|^2+\frac{1}{4}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2,$$

$$\frac{k_0}{ml}\frac{\xi^2}{\left(1+\xi^2\right)}\left|\mathrm{Re}\left\langle \frac{\xi^2}{\left(1+\xi^2\right)}\widehat{\phi},\,\widehat{\theta}\right\rangle\right| \leq C\,\xi^2\left|\widehat{\theta}\right|^2+\frac{\varepsilon_1}{2}\frac{\xi^6}{\left(1+\xi^2\right)^3}\left|\widehat{\phi}\right|^2,$$

then, we obtain

$$\begin{aligned}\frac{\xi^4}{\left(1+\xi^2\right)^2}\;\left|\widehat{y}\right|^2+\frac{\partial}{\partial t}\mathcal{F}_1(\xi,t) \;&\leq\; C\;\left|\widehat{\eta}\right|^2+C\left(\varepsilon_1\right)\xi^2\left|\widehat{\theta}\right|^2+\varepsilon_1\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2 \\ &\quad+\varepsilon_1\frac{\xi^6}{\left(1+\xi^2\right)^3}\left|\widehat{\phi}\right|^2,\end{aligned} \tag{3.10}$$

where

$$\mathcal{F}_1(\xi,t)=\frac{2\xi^2}{m\left(1+\xi^2\right)^2}\,\mathrm{Re}\left\langle \widehat{\theta},\imath\xi\,\widehat{y}\right\rangle+\frac{2\xi^2}{ml\left(1+\xi^2\right)^2}\left\langle \widehat{\eta},\imath\xi\,\widehat{\theta}\right\rangle. \tag{3.11}$$

Step2. From $(2.2)_2$, we have

$$l\,k_0\,\widehat{\phi}=\widehat{u}_t-i\xi\,\widehat{v},$$

and by using this in $(2.2)_6$, we obtain

$$l\,\widehat{\eta}_t-i\xi\widehat{u}_t\,-\left(l^{\,2}+\xi^2\right)\widehat{v}+l\gamma\,\widehat{\eta}=0. \tag{3.12}$$

Multiplying (3.12) by $\frac{i\xi}{\left(l^{\,2}+\xi^2\right)}\widehat{z}$ and using $(2.2)_3$, we get

$$\begin{aligned}0 \;&=\; \frac{l\xi}{\left(l^{\,2}+\xi^2\right)}\frac{\partial}{\partial t}\left\langle \widehat{\eta},i\,\widehat{z}\right\rangle-\frac{\xi^2}{\left(l^{\,2}+\xi^2\right)}\frac{\partial}{\partial t}\left\langle \widehat{u},\widehat{z}\right\rangle-l\left\langle \widehat{\eta},\,\frac{i\xi}{\left(l^{\,2}+\xi^2\right)}\widehat{z}_t\right\rangle+\left\langle i\xi\,\widehat{u},\,\frac{i\xi}{\left(l^{\,2}+\xi^2\right)}\widehat{z}_t\right\rangle \\ &\quad-\left\langle \widehat{v},i\xi\,\widehat{z}\right\rangle+l\gamma\left\langle \widehat{\eta},\,\frac{i\xi}{\left(l^{\,2}+\xi^2\right)}\widehat{z}\right\rangle \\ &=\; \frac{\partial}{\partial t}\left\langle \widehat{\eta},\,\frac{il\xi}{\left(l^{\,2}+\xi^2\right)}\widehat{z}\right\rangle-\frac{\partial}{\partial t}\left\langle \xi\widehat{u},\,\frac{\xi}{\left(l^{\,2}+\xi^2\right)}\widehat{z}\right\rangle+\frac{al\xi^2}{\left(l^{\,2}+\xi^2\right)}\left\langle \widehat{\eta},\,\widehat{y}\right\rangle-\frac{a\xi^3}{\left(l^{\,2}+\xi^2\right)}\left\langle i\,\widehat{u},\widehat{y}\right\rangle \\ &\quad-\left\langle \widehat{v},i\xi\,\widehat{z}\right\rangle+\frac{l\gamma\,\xi}{\left(l^{\,2}+\xi^2\right)}\left\langle \widehat{\eta},i\,\widehat{z}\right\rangle,\end{aligned}$$

then we have

$$\begin{aligned}\left\langle \widehat{v},i\xi\,\widehat{z}\right\rangle \;&=\; \frac{l\xi}{\left(l^{\,2}+\xi^2\right)}\frac{\partial}{\partial t}\left\langle \widehat{\eta},i\,\widehat{z}\right\rangle-\frac{\xi^2}{\left(l^{\,2}+\xi^2\right)}\frac{\partial}{\partial t}\left\langle \widehat{u},\widehat{z}\right\rangle+\frac{al\xi^2}{\left(l^{\,2}+\xi^2\right)}\left\langle \widehat{\eta},\,\widehat{y}\right\rangle \\ &\quad-\frac{a\xi^3}{\left(l^{\,2}+\xi^2\right)}\left\langle i\,\widehat{u},\widehat{y}\right\rangle+\frac{l\gamma\,\xi}{\left(l^{\,2}+\xi^2\right)}\left\langle \widehat{\eta},i\,\widehat{z}\right\rangle.\end{aligned} \tag{3.13}$$

Multiplying $(2.2)_4$ by $-\frac{i\xi}{a}\widehat{z}$, we get

$$-\frac{\partial}{\partial t}\left\langle \widehat{y},\,\frac{i\xi}{a}\widehat{z}\right\rangle+\xi^2\;\left|\widehat{z}\right|^2-\xi^2\;\left|\widehat{y}\right|^2+\frac{1}{a}\left\langle \widehat{v},i\xi\,\widehat{z}\right\rangle-\frac{m\xi^2}{a}\left\langle \widehat{\theta},\widehat{z}\right\rangle=0, \tag{3.14}$$

and by (3.12) and (3.14), we have

$$\begin{aligned}
&\xi^2\ \left|\widehat{z}\right|^2-\frac{\xi}{a}\frac{\partial}{\partial t}\operatorname{Re}\left\langle \widehat{y},i\,\widehat{z}\right\rangle+\frac{l\xi}{a\left(l^{\,2}+\xi^2\right)}\frac{\partial}{\partial t}\operatorname{Re}\left\langle \widehat{\eta},i\,\widehat{z}\right\rangle-\frac{\xi^2}{a\left(l^{\,2}+\xi^2\right)}\frac{\partial}{\partial t}\operatorname{Re}\left\langle \widehat{u},\widehat{z}\right\rangle\\
=\;&\xi^2\ \left|\widehat{y}\right|^2+\frac{m\xi^2}{a}\operatorname{Re}\left\langle \widehat{\theta},\widehat{z}\right\rangle-\frac{l\xi^2}{\left(l^{\,2}+\xi^2\right)}\operatorname{Re}\left\langle \widehat{\eta},\,\widehat{y}\right\rangle\\
&+\frac{\xi^3}{\left(l^{\,2}+\xi^2\right)}\operatorname{Re}\left\langle i\,\widehat{u},\widehat{y}\right\rangle-\frac{l\gamma\,\xi}{a\left(l^{\,2}+\xi^2\right)}\operatorname{Re}\left\langle \widehat{\eta},i\,\widehat{z}\right\rangle. \qquad (3.15)
\end{aligned}$$

For the case (3.1)**,** we proceed as follow:

Multiplying (3.15) by $\frac{1}{\left(1+\xi^2\right)}$, applying Young's inequality and the following estimates, for any $\varepsilon_2>0$, given by

$$\frac{m\xi^2}{a\left(1+\xi^2\right)}\left|\operatorname{Re}\left\langle \widehat{\theta},\widehat{z}\right\rangle\right|\le C\xi^2\left|\widehat{\theta}\right|^2+\frac{1}{4}\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{z}\right|^2,$$

$$\frac{l\xi^2}{\left(1+\xi^2\right)}\left|\operatorname{Re}\left\langle \frac{1}{\left(1+\xi^2\right)}\widehat{\eta},\,\widehat{y}\right\rangle\right|\le C\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{y}\right|^2+C\left|\widehat{\eta}\right|^2,$$

$$\frac{\xi^2}{\left(1+\xi^2\right)}\left|\operatorname{Re}\left\langle \frac{\xi}{\left(l^{\,2}+\xi^2\right)}i\,\widehat{u},\widehat{y}\right\rangle\right|\le C\left(\varepsilon_2\right)\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{y}\right|^2+\frac{\varepsilon_2}{2}\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{u}\right|^2,$$

$$\frac{l\gamma}{a\left(l^{\,2}+\xi^2\right)}\left|\operatorname{Re}\left\langle \frac{1}{\left(1+\xi^2\right)}\widehat{\eta},i\,\xi\widehat{z}\right\rangle\right|\le C\left|\widehat{\eta}\right|^2+\frac{1}{4}\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{z}\right|^2,$$

then, we obtain

$$\frac{\xi^2}{\left(1+\xi^2\right)}\ \left|\widehat{z}\right|^2+\frac{\partial}{\partial t}\mathcal{F}_2\left(\xi,t\right)\le C\ \left|\widehat{\eta}\right|^2+C\left(\varepsilon_2\right)\frac{\xi^2}{\left(1+\xi^2\right)}\ \left|\widehat{y}\right|^2+C\xi^2\left|\widehat{\theta}\right|^2+\varepsilon_2\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{u}\right|^2, \quad (3.16)$$

where

$$\mathcal{F}_2\left(\xi,t\right)=-\frac{2\xi}{a\left(1+\xi^2\right)}\operatorname{Re}\left\langle \widehat{y},i\,\widehat{z}\right\rangle+\frac{2l\xi}{a\left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)}\operatorname{Re}\left\langle \widehat{\eta},i\,\widehat{z}\right\rangle-\frac{2\xi^2}{a\left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)}\operatorname{Re}\left\langle \widehat{u},\widehat{z}\right\rangle. \quad (3.17)$$

For the cases $(3.2-3.3)$**,** we proceed as follow:

Multiplying (3.15) by $\frac{\xi^2}{\left(1+\xi^2\right)^2}$, applying Young's inequality and the following estimates, for any $\varepsilon_2>0$, given by

$$\begin{aligned}
&\tfrac{m\xi^4}{a\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \widehat{\theta},\widehat{z}\right\rangle\right|\le C\xi^2\left|\widehat{\theta}\right|^2+\tfrac{1}{4}\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2,\\
&\tfrac{l\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \tfrac{1}{\left(1+\xi^2\right)}\widehat{\eta},\,\widehat{y}\right\rangle\right|\le C\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2+C\left|\widehat{\eta}\right|^2,\\
&\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \tfrac{\xi}{\left(l^{\,2}+\xi^2\right)}i\,\widehat{u},\widehat{y}\right\rangle\right|\le C\left(\varepsilon_2\right)\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2+\tfrac{\varepsilon_2}{2}\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{u}\right|^2,\\
&\tfrac{l\gamma\,\xi^2}{a\left(l^{\,2}+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \tfrac{1}{\left(1+\xi^2\right)}\widehat{\eta},i\,\xi\widehat{z}\right\rangle\right|\le C\left|\widehat{\eta}\right|^2+\tfrac{1}{4}\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2,
\end{aligned}$$

then, we obtain

$$\frac{\xi^4}{\left(1+\xi^2\right)^2}\ |\widehat{z}|^2+\frac{\partial}{\partial t}\mathcal{F}_2\left(\xi,t\right)\leq C\ |\widehat{\eta}|^2+C\xi^2\left|\widehat{\theta}\right|^2+C\left(\varepsilon_2\right)\frac{\xi^4}{\left(1+\xi^2\right)^2}\ |\widehat{y}|^2+\varepsilon_2\frac{\xi^4}{\left(1+\xi^2\right)^2}\,|\widehat{u}|^2\,, \tag{3.18}$$

where

$$\mathcal{F}_2\left(\xi,t\right)=-\frac{2\xi^3}{a\left(1+\xi^2\right)^2}\operatorname{Re}\langle\widehat{y},i\,\widehat{z}\rangle+\frac{2l\xi^3}{a\left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)^2}\operatorname{Re}\langle\,\widehat{\eta},i\,\widehat{z}\rangle-\frac{2\xi^4}{a\left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)^2}\operatorname{Re}\langle\widehat{u},\widehat{z}\rangle\,. \tag{3.19}$$

For the cases $(3.4-3.5)$**,** we proceed as follow:

Multiplying (3.15) by $\frac{\xi^2}{\left(1+\xi^2\right)^2}$, applying Young's inequality and the following estimates, for any $\varepsilon_2>0$, given by

$$\begin{aligned}
&\tfrac{m\xi^4}{a\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle\,\widehat{\theta},\widehat{z}\right\rangle\right|\leq C\xi^2\left|\widehat{\theta}\right|^2+\tfrac14\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\,|\widehat{z}|^2\,,\\
&\tfrac{l\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle\,\tfrac{1}{\left(1+\xi^2\right)}\widehat{\eta},\,\widehat{y}\right\rangle\right|\leq C\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\,|\widehat{y}|^2+C\,|\widehat{\eta}|^2\,,\\
&\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\ \left\langle\,\tfrac{i\xi}{\left(l^{\,2}+\xi^2\right)}\ \widehat{u},\widehat{y}\right\rangle\right|\leq C\left(\varepsilon_2\right)\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\,|\widehat{y}|^2+\tfrac{\varepsilon_2}{2}\tfrac{\xi^4}{\left(1+\xi^2\right)^3}\,|\widehat{u}|^2\,,\\
&\tfrac{l\gamma\,\xi^2}{a\left(l^{\,2}+\xi^2\right)^2}\left|\operatorname{Re}\left\langle\,\tfrac{1}{\left(1+\xi^2\right)}\widehat{\eta},i\,\xi\widehat{z}\right\rangle\right|\leq C\,|\widehat{\eta}|^2+\tfrac14\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\,|\widehat{z}|^2\,,
\end{aligned}$$

then, we obtain

$$\frac{\xi^4}{\left(1+\xi^2\right)^2}\ |\widehat{z}|^2+\frac{\partial}{\partial t}\mathcal{F}_2\left(\xi,t\right)\leq C\ |\widehat{\eta}|^2+C\xi^2\left|\widehat{\theta}\right|^2+C\left(\varepsilon_2\right)\frac{\xi^4}{\left(1+\xi^2\right)^2}\ |\widehat{y}|^2+\varepsilon_2\frac{\xi^4}{\left(1+\xi^2\right)^3}\,|\widehat{u}|^2\,, \tag{3.20}$$

where

$$\mathcal{F}_2\left(\xi,t\right)=-\frac{2\xi^3}{a\left(1+\xi^2\right)^2}\operatorname{Re}\langle\widehat{y},i\,\widehat{z}\rangle+\frac{2l\xi^3}{a\left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)^2}\operatorname{Re}\langle\,\widehat{\eta},i\,\widehat{z}\rangle-\frac{2\xi^4}{a\left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)^2}\operatorname{Re}\langle\widehat{u},\widehat{z}\rangle\,. \tag{3.21}$$

For the cases $(3.6-3.7)$**,** we proceed as follow:

Multiplying (3.15) by $\frac{\xi^2}{\left(1+\xi^2\right)^2}$, applying Young's inequality and the following estimates, for any $\varepsilon_2>0$, given by

$$\begin{aligned}
&\tfrac{m\xi^4}{a\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle\,\widehat{\theta},\widehat{z}\right\rangle\right|\leq C\xi^2\left|\widehat{\theta}\right|^2+\tfrac14\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\,|\widehat{z}|^2\,,\\
&\tfrac{l\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle\,\tfrac{1}{\left(l^2+\xi^2\right)}\widehat{\eta},\,\widehat{y}\right\rangle\right|\leq C\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\,|\widehat{y}|^2+C\ |\widehat{\eta}|^2\,,\\
&\tfrac{\xi^4}{\left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)^2}\left|\operatorname{Re}\ \langle i\xi\,\widehat{u},\widehat{y}\rangle\right|\leq C\left(\varepsilon_2\right)\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\,|\widehat{y}|^2+\tfrac{\varepsilon_2}{2}\tfrac{\xi^6}{\left(1+\xi^2\right)^4}\,|\widehat{u}|^2\,,\\
&\tfrac{l\gamma}{a\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle\,\tfrac{\xi}{\left(l^{\,2}+\xi^2\right)}\widehat{\eta},i\xi^2\,\widehat{z}\right\rangle\leq C\ |\widehat{\eta}|^2+\tfrac14\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\,|\widehat{z}|^2\,,
\end{aligned}$$

then, we obtain

$$\frac{\xi^4}{\left(1+\xi^2\right)^2}\,|\widehat{z}|^2+\frac{\partial}{\partial t}\mathcal{F}_2(\xi,t)\leq C\,|\widehat{\eta}|^2+C\xi^2\left|\widehat{\theta}\right|^2+C(\varepsilon_2)\frac{\xi^4}{\left(1+\xi^2\right)^2}|\widehat{y}|^2+\varepsilon_2\frac{\xi^6}{\left(1+\xi^2\right)^3}|\widehat{u}|^2\,, \tag{3.22}$$

where

$$\mathcal{F}_2(\xi,t)=-\frac{2\xi^3}{a\left(1+\xi^2\right)^2}\operatorname{Re}\langle\widehat{y},i\,\widehat{z}\rangle+\frac{2l\xi^3}{a\left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)^2}\operatorname{Re}\langle\,\widehat{\eta},i\,\widehat{z}\rangle-\frac{2\xi^4}{a\left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)^2}\operatorname{Re}\langle\widehat{u},\widehat{z}\rangle\,. \tag{3.23}$$

Step3. Multiplying $(2.2)_4$ by $\widehat{v}$ and using $(2.2)_1$, we get

$$\begin{aligned}0&=\frac{\partial}{\partial t}\langle\widehat{y},\widehat{v}\rangle-\langle\widehat{y},\widehat{v}_t\rangle-a\,\langle\,i\,\xi\,\widehat{z},\widehat{v}\rangle-|\widehat{v}|^2+m\left\langle\,i\xi\,\widehat{\theta},\widehat{v}\right\rangle\\&=\frac{\partial}{\partial t}\langle\widehat{y},\widehat{v}\rangle-\langle\widehat{y},i\xi\,\widehat{u}\rangle+|\widehat{y}|^2+l\;\;\langle\widehat{y},\widehat{\eta}\rangle-a\;\langle\,i\,\xi\,\widehat{z},\widehat{v}\rangle-|\widehat{v}|^2+m\left\langle\;i\xi\,\widehat{\theta},\widehat{v}\right\rangle,\end{aligned}$$

then

$$|\widehat{v}|^2-\frac{\partial}{\partial t}\langle\widehat{y},\widehat{v}\rangle=-\langle\widehat{y},i\xi\,\widehat{u}\rangle+|\widehat{y}|^2+l\;\;\langle\widehat{y},\widehat{\eta}\rangle+m\left\langle\;i\xi\,\widehat{\theta},\widehat{v}\right\rangle-a\;\langle\,i\,\xi\,\widehat{z},\widehat{v}\rangle\,,$$

with (3.13), we deduce that

$$\begin{aligned}&|\widehat{v}|^2-\frac{\partial}{\partial t}\operatorname{Re}\langle\widehat{y},\widehat{v}\rangle+\frac{al\xi}{\left(l^{\,2}+\xi^2\right)}\frac{\partial}{\partial t}\left(\operatorname{Re}\langle\,\widehat{\eta},i\,\widehat{z}\rangle\right)-\frac{a\xi^2}{\left(l^{\,2}+\xi^2\right)}\frac{\partial}{\partial t}\operatorname{Re}\langle\widehat{u},\widehat{z}\rangle\\=\;&|\widehat{y}|^2+\frac{l}{\left(l^{\,2}+\xi^2\right)}\;\left(l^{\,2}+\left(1-a^2\right)\xi^2\right)\left(\operatorname{Re}\langle\,\widehat{\eta},\,\widehat{y}\rangle\right)+m\operatorname{Re}\left\langle\;i\xi\,\widehat{\theta},\widehat{v}\right\rangle\\&-\frac{1}{\left(l^{\,2}+\xi^2\right)}\;\left(l^{\,2}+\left(1-a^2\right)\xi^2\right)\left(\operatorname{Re}\langle i\xi\,\widehat{u},\widehat{y}\rangle\right)-\frac{al\gamma\,\xi}{\left(l^{\,2}+\xi^2\right)}\operatorname{Re}\langle\widehat{\eta},i\,\widehat{z}\rangle\,.\end{aligned} \tag{3.24}$$

For the case (3.1)**,** we proceed as follow:

Multiplying (3.24) by $\frac{\xi^2}{\left(1+\xi^2\right)}$, applying Young's inequality and the following estimates, for any $\varepsilon_3>0$, given by

$$\begin{aligned}&\tfrac{l^{\,3}\xi^2}{\left(1+\xi^2\right)}\left|\operatorname{Re}\left\langle\tfrac{1}{\left(l^{\,2}+\xi^2\right)}\,\widehat{\eta},\,\widehat{y}\right\rangle\right|\leq C\,|\widehat{\eta}|^2+C\tfrac{\xi^2}{\left(1+\xi^2\right)}\,|\widehat{y}|^2\,,\\&m\tfrac{\xi^2}{\left(1+\xi^2\right)}\left|\operatorname{Re}\left\langle\;i\xi\,\widehat{\theta},\widehat{v}\right\rangle\right|\leq C\,\xi^2\left|\widehat{\theta}\right|^2+\tfrac{1}{2}\tfrac{\xi^2}{\left(1+\xi^2\right)}\,|\widehat{v}|^2\,,\\&\tfrac{l^{\,2}\xi^2}{\left(1+\xi^2\right)}\left|\operatorname{Re}\left\langle\;\tfrac{i\xi}{\left(l^{\,2}+\xi^2\right)}\,\widehat{u},\widehat{y}\right\rangle\right|\leq C(\varepsilon_3)\,\tfrac{\xi^2}{\left(1+\xi^2\right)}\,|\widehat{y}|^2+\tfrac{\varepsilon_3}{2}\tfrac{\xi^2}{\left(1+\xi^2\right)}\,|\widehat{u}|^2\,,\\&\tfrac{al\gamma\xi^2}{\left(1+\xi^2\right)}\left|\operatorname{Re}\left\langle\;\tfrac{1}{\left(l^{\,2}+\xi^2\right)}\widehat{\eta},i\,\xi\widehat{z}\right\rangle\right|\leq C\,|\widehat{\eta}|^2+C\,\tfrac{\xi^2}{\left(1+\xi^2\right)}\,|\widehat{z}|^2\,,\end{aligned}$$

then, we obtain

$$\frac{\xi^2}{\left(1+\xi^2\right)}\,|\widehat{v}|^2+\frac{\partial}{\partial t}\mathcal{F}_3(\xi,t)\leq C\,|\widehat{\eta}|^2+C\,\xi^2\left|\widehat{\theta}\right|^2+C(\varepsilon_3)\,\frac{\xi^2}{\left(1+\xi^2\right)}\,|\widehat{y}|^2+C\,\frac{\xi^2}{\left(1+\xi^2\right)}\,|\widehat{z}|^2+\varepsilon_3\frac{\xi^2}{\left(1+\xi^2\right)}\,|\widehat{u}|^2\,, \tag{3.25}$$

where

$$\mathcal{F}_3(\xi,t) = -\frac{2\xi^2}{\left(1+\xi^2\right)} \operatorname{Re}\langle \widehat{y}, \widehat{v}\rangle + \frac{2al\xi^3}{\left(l^2+\xi^2\right)\left(1+\xi^2\right)} \operatorname{Re}\langle \widehat{\eta}, i\,\widehat{z}\rangle - \frac{2a\xi^4}{\left(l^2+\xi^2\right)\left(1+\xi^2\right)} \operatorname{Re}\langle \widehat{u}, \widehat{z}\rangle . \tag{3.26}$$

For the cases $(3.2-3.3)$**,** we proceed as follow:

Multiplying (3.24) by $\frac{\xi^4}{\left(1+\xi^2\right)^2}$, applying Young's inequality and the following estimates, for any $\varepsilon_3>0$, given by

$$\begin{aligned}
&\tfrac{l^3\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \tfrac{1}{\left(l^2+\xi^2\right)}\widehat{\eta},\,\widehat{y}\right\rangle\right| \le C\left|\widehat{\eta}\right|^2 + C\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2,\\
&m\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle i\xi\,\widehat{\theta},\widehat{v}\right\rangle\right| \le C\,\xi^2\left|\widehat{\theta}\right|^2 + \tfrac{1}{2}\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{v}\right|^2,\\
&\tfrac{l^2\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \tfrac{i\xi}{\left(l^2+\xi^2\right)}\,\widehat{u},\widehat{y}\right\rangle\right| \le C\left(\varepsilon_3\right)\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2 + \tfrac{\varepsilon_3}{2}\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{u}\right|^2,\\
&\tfrac{al\gamma\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \tfrac{i\,\xi}{\left(l^2+\xi^2\right)}\widehat{\eta},\widehat{z}\right\rangle\right| \le C\left|\widehat{\eta}\right|^2 + C\,\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2,
\end{aligned}$$

then, we obtain

$$\begin{aligned}
\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{v}\right|^2 + \frac{\partial}{\partial t}\mathcal{F}_3(\xi,t) \;\le\; & C\left|\widehat{\eta}\right|^2 + C\,\xi^2\left|\widehat{\theta}\right|^2 + C\left(\varepsilon_3\right)\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2 \\
& + C\,\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2 + \varepsilon_3\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{u}\right|^2,
\end{aligned} \tag{3.27}$$

where

$$\mathcal{F}_3(\xi,t) = -\frac{2\xi^4}{\left(1+\xi^2\right)^2} \operatorname{Re}\langle \widehat{y}, \widehat{v}\rangle + \frac{2al\xi^5}{\left(l^2+\xi^2\right)\left(1+\xi^2\right)^2} \operatorname{Re}\langle \widehat{\eta}, i\,\widehat{z}\rangle - \frac{2a\xi^6}{\left(l^2+\xi^2\right)\left(1+\xi^2\right)^2} \operatorname{Re}\langle \widehat{u}, \widehat{z}\rangle . \tag{3.28}$$

For the cases $(3.4-3.5)$**,** we proceed as follow:

Multiplying (3.24) by $\frac{\xi^4}{\left(1+\xi^2\right)^3}$, applying Young's inequality and the following estimates, for any $\varepsilon_3>0$, given by

$$\begin{aligned}
&\tfrac{l^3\xi^4}{\left(1+\xi^2\right)^3}\left|\operatorname{Re}\left\langle \tfrac{\left(l^2+\left(1-a^2\right)\xi^2\right)}{\left(l^2+\xi^2\right)}\,\widehat{\eta},\,\widehat{y}\right\rangle\right| \le C\left|\widehat{\eta}\right|^2 + C\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2,\\
&m\tfrac{\xi^4}{\left(1+\xi^2\right)^3}\left|\operatorname{Re}\left\langle i\xi\,\widehat{\theta},\widehat{v}\right\rangle\right| \le C\,\xi^2\left|\widehat{\theta}\right|^2 + \tfrac{1}{2}\tfrac{\xi^4}{\left(1+\xi^2\right)^3}\left|\widehat{v}\right|^2,\\
&\tfrac{l^2\xi^4}{\left(1+\xi^2\right)^3}\left|\operatorname{Re}\left\langle \tfrac{i\xi}{\left(l^2+\xi^2\right)}\,\widehat{u},\widehat{y}\right\rangle\right| \le C\left(\varepsilon_3\right)\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2 + \tfrac{\varepsilon_3}{2}\tfrac{\xi^4}{\left(1+\xi^2\right)^3}\left|\widehat{u}\right|^2,\\
&\tfrac{al\gamma\xi^4}{\left(1+\xi^2\right)^3}\left|\operatorname{Re}\left\langle \tfrac{i\,\xi}{\left(l^2+\xi^2\right)}\widehat{\eta},\widehat{z}\right\rangle\right| \le C\left|\widehat{\eta}\right|^2 + C\,\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2,
\end{aligned}$$

then, we obtain

$$\begin{aligned}
\frac{\xi^4}{\left(1+\xi^2\right)^3}\left|\widehat{v}\right|^2 + \frac{\partial}{\partial t}\mathcal{F}_3(\xi,t) \;\le\; & C\left|\widehat{\eta}\right|^2 + C\,\xi^2\left|\widehat{\theta}\right|^2 + C\left(\varepsilon_3\right)\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2 \\
& + C\,\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2 + \varepsilon_3\frac{\xi^4}{\left(1+\xi^2\right)^3}\left|\widehat{u}\right|^2,
\end{aligned} \tag{3.29}$$

where

$$\mathcal{F}_3(\xi,t) = -\frac{2\xi^4}{\left(1+\xi^2\right)^3}\operatorname{Re}\langle\widehat{y},\widehat{v}\rangle + \frac{2al\xi^5}{\left(l^2+\xi^2\right)\left(1+\xi^2\right)^3}\operatorname{Re}\langle\widehat{\eta}, i\,\widehat{z}\rangle - \frac{2a\xi^6}{\left(l^2+\xi^2\right)\left(1+\xi^2\right)^3}\operatorname{Re}\langle\widehat{u},\widehat{z}\rangle . \tag{3.30}$$

For the cases $(3.6-3.7)$**,** we proceed as follow:

Multiplying (3.24) by $\frac{\xi^6}{\left(1+\xi^2\right)^4}$, applying Young's inequality and the following estimates, for any $\varepsilon_3>0$, given by

$$\begin{aligned}
&\tfrac{l\xi^6}{\left(1+\xi^2\right)^4}\left|\tfrac{\left(l^2+\left(1-a^2\right)\xi^2\right)}{\left(l^2+\xi^2\right)}\left(\operatorname{Re}\langle\widehat{\eta},\,\widehat{y}\rangle\right)\right| \le C\left|\widehat{\eta}\right|^2 + C\,\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2,\\
&\tfrac{m\xi^6}{\left(1+\xi^2\right)^4}\left|\operatorname{Re}\left\langle i\xi\,\widehat{\theta},\widehat{v}\right\rangle\right| \le C\,\xi^2\left|\widehat{\theta}\right|^2 + \tfrac{1}{2}\tfrac{\xi^6}{\left(1+\xi^2\right)^4}\left|\widehat{v}\right|^2,\\
&\tfrac{\xi^6}{\left(1+\xi^2\right)^4}\left|\tfrac{\left(l^2+\left(1-a^2\right)\xi^2\right)}{\left(l^2+\xi^2\right)}\left(\operatorname{Re}\langle\widehat{u}, i\xi\,\widehat{y}\rangle\right)\right| \le C\left(\varepsilon_3\right)\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2 + \tfrac{\varepsilon_3}{2}\tfrac{\xi^6}{\left(1+\xi^2\right)^4}\left|\widehat{u}\right|^2,\\
&\tfrac{al\gamma\xi^6}{\left(1+\xi^2\right)^4}\left|\operatorname{Re}\left\langle\tfrac{\xi}{\left(l^2+\xi^2\right)}\widehat{\eta}, i\,\widehat{z}\right\rangle\right| \le C\left|\widehat{\eta}\right|^2 + C\,\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2,
\end{aligned}$$

then, we obtain

$$\begin{aligned}
\frac{\xi^4}{\left(1+\xi^2\right)^3}\left|\widehat{v}\right|^2 + \frac{\partial}{\partial t}\mathcal{F}_3(\xi,t) \;&\le\; C\left|\widehat{\eta}\right|^2 + C\,\xi^2\left|\widehat{\theta}\right|^2 + \varepsilon_3\frac{\xi^6}{\left(1+\xi^2\right)^4}\left|\widehat{u}\right|^2 \\
&\quad + C\left(\varepsilon_3\right)\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2 + C\,\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2,
\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}
\mathcal{F}_3(\xi,t) \;&=\; -\frac{2\xi^6}{\left(1+\xi^2\right)^4}\frac{\partial}{\partial t}\operatorname{Re}\langle\widehat{y},\widehat{v}\rangle + \frac{2al\xi^7}{\left(l^2+\xi^2\right)\left(1+\xi^2\right)^4}\frac{\partial}{\partial t}\operatorname{Re}\langle\widehat{\eta}, i\,\widehat{z}\rangle \\
&\quad - \frac{2a\xi^8}{\left(l^2+\xi^2\right)\left(1+\xi^2\right)^4}\frac{\partial}{\partial t}\operatorname{Re}\langle\widehat{u},\widehat{z}\rangle .
\end{aligned} \tag{3.32}$$

Step4. Multiplying $(2.2)_4$ by $i\xi\widehat{\phi}$, and using $(2.2)_5$, we get

$$\begin{aligned}
0 \;&=\; \frac{\partial}{\partial t}\left\langle\widehat{y}, i\xi\,\widehat{\phi}\right\rangle - \left\langle\widehat{y}, i\xi\,\widehat{\phi}_t\right\rangle - a\left(i\,\xi\,\widehat{z}, i\xi\,\widehat{\phi}\right) - \left\langle\widehat{v}, i\xi\,\widehat{\phi}\right\rangle + m\left\langle i\xi\,\widehat{\theta}, i\xi\,\widehat{\phi}\right\rangle \\
&=\; \frac{\partial}{\partial t}\left\langle\widehat{y}, i\xi\,\widehat{\phi}\right\rangle + k_0\,\xi^2\langle\widehat{y},\widehat{\eta}\rangle + l\,k_0\langle\widehat{y}, i\xi\,\widehat{u}\rangle - a\,\xi^2\left\langle\,\widehat{z},\widehat{\phi}\right\rangle - \left\langle\widehat{v}, i\xi\,\widehat{\phi}\right\rangle + m\,\xi^2\left\langle\widehat{\theta},\,\widehat{\phi}\right\rangle,
\end{aligned}$$

then, we obtain

$$\left\langle\widehat{v}, i\xi\,\widehat{\phi}\right\rangle = \frac{\partial}{\partial t}\left\langle\widehat{y}, i\xi\,\widehat{\phi}\right\rangle + k_0\,\xi^2\langle\widehat{y},\widehat{\eta}\rangle + l\,k_0\langle\widehat{y}, i\xi\,\widehat{u}\rangle - a\,\xi^2\left\langle\,\widehat{z},\widehat{\phi}\right\rangle + m\,\xi^2\left\langle\widehat{\theta},\,\widehat{\phi}\right\rangle . \tag{3.33}$$

Multiplying $(2.2)_2$ by $-i\xi\,\widehat{v}$ and using $(2.2)_1$, we have

$$\begin{aligned}
0 \;&=\; -\frac{\partial}{\partial t}\langle\widehat{u}, i\xi\,\widehat{v}\rangle + \langle\widehat{u}, i\xi\,\widehat{v}_t\rangle + \xi^2\left|\widehat{v}\right|^2 - l\,k_0\left\langle i\xi\,\widehat{\phi},\widehat{v}\right\rangle \\
&=\; -\frac{\partial}{\partial t}\langle\widehat{u}, i\xi\,\widehat{v}\rangle - \xi^2\left|\widehat{u}\right|^2 - \langle\widehat{u}, i\xi\,\widehat{y}\rangle - l\,\langle\widehat{u}, i\xi\,\widehat{\eta}\rangle + \xi^2\left|\widehat{v}\right|^2 - l\,k_0\left\langle i\xi\,\widehat{\phi},\widehat{v}\right\rangle,
\end{aligned} \tag{3.34}$$

by using (3.33) and (3.34), we obtain

$$\begin{aligned} & \xi^2 |\widehat{u}|^2 + \frac{\partial}{\partial t} \operatorname{Re} \langle \widehat{u}, i\xi\, \widehat{v} \rangle + l\, k_0 \frac{\partial}{\partial t} \left(\operatorname{Re} \left\langle \widehat{y}, i\xi\, \widehat{\phi} \right\rangle \right) \\ = \; & \xi^2 |\widehat{v}|^2 + \left(1 - l^2 k_0^2\right) \operatorname{Re} \langle i\xi \widehat{u},\, \widehat{y} \rangle + l\, \operatorname{Re} \langle i\xi \widehat{u},\, \widehat{\eta} \rangle - l\, k_0^2\, \xi^2 \operatorname{Re} \langle \widehat{y}, \widehat{\eta} \rangle \\ & + al\, k_0\, \xi^2 \operatorname{Re} \left\langle\, \widehat{z}, \widehat{\phi} \right\rangle - l\, k_0 m\, \xi^2 \operatorname{Re} \left\langle \widehat{\theta},\, \widehat{\phi} \right\rangle . \end{aligned} \tag{3.35}$$

Multiplying $(2.2)_3$ by $\widehat{u}$ and using $(2.2)_2$, we get

$$\begin{aligned} 0 \; &= \; \frac{\partial}{\partial t} \langle \widehat{z}, \widehat{u} \rangle - \langle \widehat{z}, \widehat{u}_t \rangle - a \; \langle i\xi\, \widehat{y}, \widehat{u} \rangle \\ &= \; \frac{\partial}{\partial t} \langle \widehat{z}, \widehat{u} \rangle + \langle i\xi \widehat{z},\, \widehat{v} \rangle - l\, k_0 \left\langle \widehat{z},\, \widehat{\phi} \right\rangle + a \; \langle \widehat{y}, i\xi \widehat{u} \rangle , \end{aligned}$$

and by using (3.13), we have

$$\begin{aligned} 0 \; = \; & \frac{\partial}{\partial t} \operatorname{Re} \langle \widehat{z}, \widehat{u} \rangle - l\, k_0 \operatorname{Re} \left\langle \widehat{z},\, \widehat{\phi} \right\rangle + \frac{al^{\,2}}{\left(l^{\,2} + \xi^2\right)} \operatorname{Re} \; \langle i\xi\, \widehat{u}, \widehat{y} \rangle + \frac{l\gamma\, \xi}{\left(l^{\,2} + \xi^2\right)} \operatorname{Re} \langle \widehat{\eta}, i\, \widehat{z} \rangle \\ & + \frac{l\xi}{\left(l^{\,2} + \xi^2\right)} \frac{\partial}{\partial t} \operatorname{Re} \langle \, \widehat{\eta}, i\, \widehat{z} \rangle - \frac{\xi^2}{\left(l^{\,2} + \xi^2\right)} \frac{\partial}{\partial t} \operatorname{Re} \langle \widehat{u}, \widehat{z} \rangle + \frac{al\xi^2}{\left(l^{\,2} + \xi^2\right)} \operatorname{Re} \langle \, \widehat{\eta},\, \widehat{y} \rangle , \end{aligned}$$

then, we deduce that

$$\begin{aligned} \frac{al^{\,2}}{\left(l^{\,2} + \xi^2\right)} \operatorname{Re} \; \langle i\xi\, \widehat{u}, \widehat{y} \rangle \; = \; & \frac{l^{\,2}}{\left(l^{\,2} + \xi^2\right)} \frac{\partial}{\partial t} \operatorname{Re} \langle \widehat{u}, \widehat{z} \rangle + l\, k_0 \operatorname{Re} \left\langle \widehat{z},\, \widehat{\phi} \right\rangle - \frac{l\gamma\, \xi}{\left(l^{\,2} + \xi^2\right)} \operatorname{Re} \langle \widehat{\eta}, i\, \widehat{z} \rangle \\ & - \frac{l\xi}{\left(l^{\,2} + \xi^2\right)} \frac{\partial}{\partial t} \operatorname{Re} \langle \, \widehat{\eta}, i\, \widehat{z} \rangle - \frac{al\xi^2}{\left(l^{\,2} + \xi^2\right)} \operatorname{Re} \langle \, \widehat{\eta},\, \widehat{y} \rangle , \end{aligned}$$

then, we have

$$\begin{aligned} l\, k_0 \operatorname{Re} \left\langle \widehat{z},\, \widehat{\phi} \right\rangle \; = \; & \frac{al^{\,2}}{\left(l^{\,2} + \xi^2\right)} \operatorname{Re} \; \langle i\xi\, \widehat{u}, \widehat{y} \rangle + \frac{l^{\,2}}{\left(l^{\,2} + \xi^2\right)} \frac{\partial}{\partial t} \operatorname{Re} \langle \widehat{z}, \widehat{u} \rangle + \\ & \frac{l\xi}{\left(l^{\,2} + \xi^2\right)} \frac{\partial}{\partial t} \left(\operatorname{Re} \langle \, \widehat{\eta}, i\, \widehat{z} \rangle \right) + \frac{la\xi^2}{\left(l^{\,2} + \xi^2\right)} \operatorname{Re} \langle \, \widehat{\eta},\, \widehat{y} \rangle \\ & + \frac{l\gamma\, \xi}{\left(l^{\,2} + \xi^2\right)} \operatorname{Re} \langle \widehat{\eta}, i\, \widehat{z} \rangle . \end{aligned} \tag{3.36}$$

For the case (3.1)**,** we proceed as follow:

Multiplying (3.35) by $\frac{1}{\left(1+\xi^2\right)}$, applying Young's inequality and the following estimates, for any $\varepsilon_4 > 0$, given by

$$\begin{aligned} & \tfrac{l}{\left(1+\xi^2\right)} \; \left| \operatorname{Re} \langle i\xi \widehat{u},\, \widehat{\eta} \rangle \right| \leq C \left| \widehat{\eta} \right|^2 + \tfrac{1}{2} \tfrac{\xi^2}{\left(1+\xi^2\right)} \left| \widehat{u} \right|^2 , \\ & \tfrac{l^2 k_0^2\, \xi^2}{\left(1+\xi^2\right)} \left| \operatorname{Re} \langle \widehat{y}, \widehat{\eta} \rangle \right| \leq C \left| \widehat{\eta} \right|^2 + C \tfrac{\xi^2}{\left(1+\xi^2\right)} \left| \widehat{y} \right|^2 , \\ & \tfrac{al\, k_0 \xi^2}{\left(1+\xi^2\right)} \left| \operatorname{Re} \left\langle\, \widehat{z}, \widehat{\phi} \right\rangle \right| \leq C\left(\varepsilon_4\right) \tfrac{\xi^2}{\left(1+\xi^2\right)} \left| \widehat{z} \right|^2 + \tfrac{\varepsilon_4}{4} \tfrac{\xi^2}{\left(1+\xi^2\right)} \left| \widehat{\phi} \right|^2 , \\ & \tfrac{l\, k_0 m\, \xi^2}{\left(1+\xi^2\right)} \left| \operatorname{Re} \left\langle \widehat{\theta},\, \widehat{\phi} \right\rangle \right| \leq C\left(\varepsilon_4\right) \xi^2 \left| \widehat{\theta} \right|^2 + \tfrac{\varepsilon_4}{4} \tfrac{\xi^2}{\left(1+\xi^2\right)} \left| \widehat{\phi} \right|^2 , \end{aligned}$$

then, we have

$$\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{u}\right|^2+\frac{\partial}{\partial t}\mathcal{F}_4\left(\xi,t\right)\le C\left|\widehat{\eta}\right|^2+C\left(\varepsilon_4\right)\xi^2\left|\widehat{\theta}\right|^2+C\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{y}\right|^2$$
$$+C\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{v}\right|^2+\varepsilon_4\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{\phi}\right|^2, \tag{3.37}$$

where

$$\mathcal{F}_4\left(\xi,t\right)=\frac{2}{\left(1+\xi^2\right)}\operatorname{Re}\left\langle \widehat{u},i\xi\,\widehat{v}\right\rangle+\frac{2l\,k_0}{\left(1+\xi^2\right)}\operatorname{Re}\left\langle \widehat{y},i\xi\,\widehat{\phi}\right\rangle. \tag{3.38}$$

For the remaining cases, we proceed as follow:

Using $(2.2)_5$, we deduce

$$\begin{aligned}\operatorname{Re}\left\langle i\xi\,\widehat{\phi},\,\widehat{\eta}_t\right\rangle &= \frac{\partial}{\partial t}\operatorname{Re}\left\langle i\xi\widehat{\phi},\,\widehat{\eta}\right\rangle+\operatorname{Re}\left\langle \widehat{\phi}_t,\,i\xi\widehat{\eta}\right\rangle \\ &= \frac{\partial}{\partial t}\operatorname{Re}\left\langle i\xi\widehat{\phi},\,\widehat{\eta}\right\rangle+\operatorname{Re}\left\langle k_0\,i\,\xi\,\widehat{\eta}-l\,k_0\widehat{u},\,i\xi\widehat{\eta}\right\rangle; \\ &= \frac{\partial}{\partial t}\operatorname{Re}\left\langle i\xi\widehat{\phi},\,\widehat{\eta}\right\rangle+k_0\xi^2\left|\widehat{\eta}\right|^2+lk_0\operatorname{Re}\left\langle i\xi\,\widehat{u},\widehat{\eta}\right\rangle.\end{aligned} \tag{3.39}$$

Using $(2.2)_5$ and $(2.2)_4$, we have

$$\begin{aligned}l^2\operatorname{Re}\left\langle i\xi\,\widehat{u},\widehat{y}\right\rangle &= -l^2\operatorname{Re}\left\langle \widehat{u},i\xi\widehat{y}\right\rangle \\ &= \frac{l}{k_0}\operatorname{Re}\left\langle \left(\widehat{\phi}_t-\,k_0\,i\,\xi\,\widehat{\eta}\right),i\xi\widehat{y}\right\rangle \\ &= \frac{l}{k_0}\frac{\partial}{\partial t}\operatorname{Re}\left\langle \widehat{\phi},i\xi\widehat{y}\right\rangle+\frac{l}{k_0}\operatorname{Re}\left\langle i\xi\,\widehat{\phi},\widehat{y}_t\right\rangle-l\xi^2\operatorname{Re}\left\langle \,\widehat{\eta},\widehat{y}\right\rangle \\ &= \frac{l}{k_0}\frac{\partial}{\partial t}\operatorname{Re}\left\langle \widehat{\phi},i\xi\widehat{y}\right\rangle+\frac{l\,a}{k_0}\xi^2\operatorname{Re}\left\langle \widehat{\phi},\,\widehat{z}\right\rangle \\ &\quad -\frac{lm}{k_0}\xi^2\operatorname{Re}\left\langle \widehat{\phi},\;\widehat{\theta}\right\rangle-l\xi^2\operatorname{Re}\left\langle \,\widehat{\eta},\widehat{y}\right\rangle+\frac{l}{k_0}\operatorname{Re}\left\langle i\xi\,\widehat{\phi},\,\widehat{v}\right\rangle,\end{aligned}$$

and by using $(2.2)_6$, we get

$$\begin{aligned}l^2\operatorname{Re}\left\langle i\xi\,\widehat{u},\widehat{y}\right\rangle &= \frac{l}{k_0}\frac{\partial}{\partial t}\operatorname{Re}\left\langle \widehat{\phi},i\xi\widehat{y}\right\rangle+\frac{l\,a}{k_0}\xi^2\operatorname{Re}\left\langle \widehat{\phi},\,\widehat{z}\right\rangle \\ &\quad -\frac{lm}{k_0}\xi^2\operatorname{Re}\left\langle \widehat{\phi},\;\widehat{\theta}\right\rangle-l\xi^2\operatorname{Re}\left\langle \,\widehat{\eta},\widehat{y}\right\rangle+\frac{1}{k_0}\operatorname{Re}\left\langle i\xi\,\widehat{\phi},\,\widehat{\eta}_t-k_0\,i\xi\,\widehat{\phi}+\gamma\,\widehat{\eta}\right\rangle \\ &= \frac{l}{k_0}\frac{\partial}{\partial t}\operatorname{Re}\left\langle \widehat{\phi},i\xi\widehat{y}\right\rangle+\frac{l\,a}{k_0}\xi^2\operatorname{Re}\left\langle \widehat{\phi},\,\widehat{z}\right\rangle-\frac{lm}{k_0}\xi^2\operatorname{Re}\left\langle \widehat{\phi},\;\widehat{\theta}\right\rangle-l\xi^2\operatorname{Re}\left\langle \,\widehat{\eta},\widehat{y}\right\rangle \\ &\quad -\xi^2\left|\widehat{\phi}\right|^2+\frac{\gamma}{k_0}\operatorname{Re}\left\langle i\xi\,\widehat{\phi},\;\widehat{\eta}\right\rangle+\frac{1}{k_0}\operatorname{Re}\left\langle i\xi\,\widehat{\phi},\,\widehat{\eta}_t\right\rangle,\end{aligned}$$

and finally by (3.39), we obtain

$$
\begin{aligned}
\operatorname{Re}\langle i\xi\,\widehat{u},\widehat{y}\rangle \;=\;& \frac{1}{lk_0}\frac{\partial}{\partial t}\operatorname{Re}\left\langle \widehat{\phi}, i\xi\widehat{y}\right\rangle + \frac{1}{l^{2}k_0}\frac{\partial}{\partial t}\operatorname{Re}\left\langle i\xi\widehat{\phi},\widehat{\eta}\right\rangle \\
&+\frac{a}{lk_0}\xi^2\operatorname{Re}\left\langle \widehat{\phi},\widehat{z}\right\rangle - \frac{m}{lk_0}\xi^2\operatorname{Re}\left\langle \widehat{\phi},\ \widehat{\theta}\right\rangle \\
&-\frac{1}{l}\xi^2\operatorname{Re}\langle\widehat{\eta},\widehat{y}\rangle + \frac{\gamma}{l^{2}k_0}\operatorname{Re}\left\langle i\xi\,\widehat{\phi},\ \widehat{\eta}\right\rangle \\
&-\frac{1}{l^{2}}\xi^2\left|\widehat{\phi}\right|^2 + \frac{1}{l^{2}}\xi^2\,|\widehat{\eta}|^2 + \frac{1}{l}\operatorname{Re}\langle i\xi\,\widehat{u},\widehat{\eta}\rangle
\end{aligned}
\tag{3.40}
$$

by using (3.35) and (3.40)

$$
\begin{aligned}
&\xi^2|\widehat{u}|^2 + \frac{(1-l^2k_0^2)}{l^{2}}\xi^2\left|\widehat{\phi}\right|^2 + \frac{\partial}{\partial t}\operatorname{Re}\langle\widehat{u}, i\xi\,\widehat{v}\rangle - l\,k_0\frac{\partial}{\partial t}\left(\operatorname{Re}\left\langle i\xi\widehat{y},\ \widehat{\phi}\right\rangle\right) \qquad (3.41)\\
&-\frac{(1-l^2k_0^2)}{lk_0}\frac{\partial}{\partial t}\operatorname{Re}\left\langle\widehat{\phi}, i\xi\widehat{y}\right\rangle - \frac{(1-l^2k_0^2)}{l^{2}k_0}\frac{\partial}{\partial t}\left(\operatorname{Re}\left\langle i\xi\widehat{\phi},\ \widehat{\eta}\right\rangle\right)\\
=\;& \xi^2|\widehat{v}|^2 + \frac{(1-l^2k_0^2)}{l^{2}}\xi^2\,|\widehat{\eta}|^2 + \frac{1}{l}\left(1+l^2\left(1-k_0^2\right)\right)\operatorname{Re}\langle i\xi\widehat{u},\ \widehat{\eta}\rangle - \frac{1}{l}\,\xi^2\operatorname{Re}\langle\widehat{y},\widehat{\eta}\rangle\\
&+\frac{a}{lk_0}\xi^2\operatorname{Re}\left\langle\widehat{\phi},\ \widehat{z}\right\rangle - \frac{m}{lk_0}\xi^2\operatorname{Re}\left\langle\widehat{\phi},\ \widehat{\theta}\right\rangle + \frac{\gamma\,(1-l^2k_0^2)}{l^{2}k_0}\operatorname{Re}\left\langle i\xi\,\widehat{\phi},\ \widehat{\eta}\right\rangle.
\end{aligned}
$$

Using (3.55), we get

$$
\begin{aligned}
&\left(\frac{(1+l^{2}\,(1-k_0^2))+\xi^2}{\left(l^{2}+\xi^2\right)}\right)\xi^2\,|\widehat{u}|^2 + \frac{(1-l^2k_0^2)}{l\,k_0\,\left(l^{2}+\xi^2\right)}\xi^2\frac{\partial}{\partial t}\left(\operatorname{Re}\left\langle\widehat{u},\widehat{\phi}\right\rangle\right)\\
&+\frac{\partial}{\partial t}\operatorname{Re}\langle\widehat{u}, i\xi\,\widehat{v}\rangle + \frac{\xi^2}{lk_0\left(1+\xi^2\right)^2}\frac{\partial}{\partial t}\left(\operatorname{Re}\left\langle\widehat{y}, i\xi\,\widehat{\phi}\right\rangle\right)\\
&-\frac{(1-l^2k_0^2)}{k_0\left(l^{2}+\xi^2\right)}\frac{\partial}{\partial t}\left(\operatorname{Re}\left\langle i\xi\widehat{\phi},\ \widehat{\eta}\right\rangle\right)\\
=\;& \xi^2|\widehat{v}|^2 + \frac{(1-l^2k_0^2)}{\left(l^{2}+\xi^2\right)}\xi^2\,|\widehat{\eta}|^2 + \frac{\gamma\,(1-l^2k_0^2)}{l^{2}k_0\left(l^{2}+\xi^2\right)}\operatorname{Re}\left\langle\widehat{\eta}, i\xi\,\widehat{\phi}\right\rangle - \frac{1}{l}\,\xi^2\operatorname{Re}\langle\widehat{y},\widehat{\eta}\rangle\\
&+\frac{\left(l^{2}\,(1-l^2k_0^2+2l^2)+(l^2+l^2k_0^2-1)\,\xi^2\right)}{l\left(l^{2}+\xi^2\right)}\operatorname{Re}\langle\widehat{\eta}, i\xi\widehat{u}\,\rangle\\
&+\frac{a}{lk_0}\xi^2\operatorname{Re}\left\langle\widehat{\phi},\ \widehat{z}\right\rangle - \frac{m}{lk_0}\xi^2\operatorname{Re}\left\langle\widehat{\phi},\ \widehat{\theta}\right\rangle.
\end{aligned}
\tag{3.42}
$$

For the case 3.2**,** we proceed as follow:

Multiplying (3.41) by $\frac{\xi^2}{\left(1+\xi^2\right)^2}$, applying Young's inequality and the following estimates, for any $\varepsilon_4 > 0$, given by

$$\begin{array}{l}
\frac{\left(1+l^2\left(1-k_0^2\right)\right)\xi^2}{l\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle i\xi\widehat{u},\ \widehat{\eta}\right\rangle\right| \leq C\left|\widehat{\eta}\right|^2+\frac{1}{2}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{u}\right|^2, \\
\frac{\xi^4}{l\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \widehat{y},\widehat{\eta}\right\rangle\right| \leq C\left|\widehat{\eta}\right|^2+C\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2, \\
\frac{a\xi^4}{lk_0\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \widehat{\phi},\ \widehat{z}\right\rangle\right| \leq C\left(\varepsilon_4\right)\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2+\frac{\varepsilon_4}{6}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2, \\
\frac{m}{lk_0}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \widehat{\phi},\ \widehat{\theta}\right\rangle\right| \leq C\left(\varepsilon_4\right)\xi^2\left|\widehat{\theta}\right|^2+\frac{\varepsilon_4}{6}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2, \\
\frac{\gamma\left(1-l^2k_0^2\right)\xi^2}{l^2k_0\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle i\xi\widehat{\phi},\ \widehat{\eta}\right\rangle\right| \leq C\left(\varepsilon_4\right)\left|\widehat{\eta}\right|^2+\frac{\varepsilon_4}{6}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2,
\end{array}$$

then, we have

$$\begin{aligned}
&\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{u}\right|^2+\frac{\partial}{\partial t}\mathcal{F}_4\left(\xi,t\right) \qquad (3.43)\\
\leq\ & C\left(\varepsilon_4\right)\left|\widehat{\eta}\right|^2+C\left(\varepsilon_4\right)\xi^2\left|\widehat{\theta}\right|^2+C\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2\\
&+C\left(\varepsilon_4\right)\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2+C\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{v}\right|^2+\varepsilon_4\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}_4\left(\xi,t\right) =\ & \frac{2\xi^2}{\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle \widehat{u},i\xi\,\widehat{v}\right\rangle+\frac{2l\,k_0\xi^2}{\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle \widehat{y},i\xi\,\widehat{\phi}\right\rangle \qquad (3.44)\\
&-\frac{2\left(1-l^2k_0^2\right)\xi^2}{lk_0\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle \widehat{\phi},i\xi\widehat{y}\right\rangle-\frac{2\left(1-l^2k_0^2\right)\xi^2}{l^2k_0\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle i\xi\widehat{\phi},\ \widehat{\eta}\right\rangle.
\end{aligned}$$

For the case (3.3)**,** we proceed as follow:

Multiplying (3.42) by $\frac{\xi^2}{\left(1+\xi^2\right)^2}$, applying Young's inequality and the following estimates, for any $\varepsilon_4 > 0$, given by

$$\begin{array}{l}
\frac{\gamma\xi^2}{\alpha_1 l^2k_0\left(1+\xi^2\right)^2}\left|\left(1-l^2k_0^2\right)\operatorname{Re}\left\langle \frac{\widehat{\eta}}{\left(l^2+\xi^2\right)},i\xi\,\widehat{\phi}\right\rangle\right| \leq C\left(\varepsilon_4\right)\left|\widehat{\eta}\right|^2+\frac{\varepsilon_4}{6}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2, \\
\frac{\xi^4}{\alpha_1 l\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \widehat{y},\widehat{\eta}\right\rangle\right| \leq C\left|\widehat{\eta}\right|^2+C\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2, \\
\frac{\xi^2}{\alpha_1\left(1+\xi^2\right)^2}\left|\frac{\left(l^2\left(1-l^2k_0^2+2l^2\right)+\left(l^2+l^2k_0^2-1\right)\xi^2\right)}{l\left(l^2+\xi^2\right)}\operatorname{Re}\left\langle \widehat{\eta},i\xi\widehat{u}\right\rangle\right| \leq C\left|\widehat{\eta}\right|^2+\frac{1}{2}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{u}\right|^2, \\
\frac{a}{\alpha_1 lk_0}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \widehat{\phi},\ \widehat{z}\right\rangle\right| \leq C\left(\varepsilon_4\right)\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2+\frac{\varepsilon_4}{6}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2, \\
\frac{m}{\alpha_1 lk_0}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \widehat{\phi},\ \widehat{\theta}\right\rangle\right| \leq C\left(\varepsilon_4\right)\xi^2\left|\widehat{\theta}\right|^2+\frac{\varepsilon_4}{6}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2,
\end{array}$$

where $\alpha_1 = \min\left(1, \left(1 + l^{\,2}\left(1 - k_0^2\right)\right)\right)$. Then we have

$$\begin{aligned}\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{u}\right|^2 + \frac{\partial}{\partial t}\mathcal{F}_4\left(\xi,t\right) &\leq C\left(\varepsilon_4\right)\left|\widehat{\eta}\right|^2 + C\left(\varepsilon_4\right)\xi^2\left|\widehat{\theta}\right|^2 + C\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2 \\ &\quad + C\left(\varepsilon_4\right)\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2 + C\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{v}\right|^2 + \varepsilon_4\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2, \end{aligned} \tag{3.45}$$

where

$$\begin{aligned}\mathcal{F}_4\left(\xi,t\right) &= \frac{2\left[1 - l^2k_0^2\right]}{\alpha_1 l\, k_0\,\left(l^{\,2}+\xi^2\right)}\frac{\xi^4}{\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle \widehat{u}, \widehat{\phi}\right\rangle \\ &\quad + \frac{2\xi^2}{\alpha_1\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle \widehat{u}, i\xi\,\widehat{v}\right\rangle + \frac{2\xi^4}{\alpha_1 l k_0\left(1+\xi^2\right)^4}\operatorname{Re}\left\langle \widehat{y}, i\xi\,\widehat{\phi}\right\rangle \\ &\quad - \frac{2\left[1 - l^2k_0^2\right]\xi^2}{\alpha_1 k_0\left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle\, i\xi\widehat{\phi},\, \widehat{\eta}\right\rangle. \end{aligned} \tag{3.46}$$

For the case (3.4)**,** we proceed as follow:

Multiplying (3.41) by $\frac{\xi^2}{\left(1+\xi^2\right)^3}$, applying Young's inequality and the following estimates, for any $\varepsilon_4 > 0$, given by

$$\begin{aligned}&\tfrac{\left[1+l^2\left(1-k_0^2\right)\right]\xi^2}{l\left(1+\xi^2\right)^3}\left|\operatorname{Re}\left\langle i\xi\widehat{u},\, \widehat{\eta}\right\rangle\right| \leq C\left|\widehat{\eta}\right|^2 + \tfrac{1}{2}\tfrac{\xi^4}{\left(1+\xi^2\right)^3}\left|\widehat{u}\right|^2, \\ &\tfrac{\xi^4}{l\left(1+\xi^2\right)^3}\;\left|\operatorname{Re}\left\langle \widehat{y}, \widehat{\eta}\right\rangle\right| \leq C\left|\widehat{\eta}\right|^2 + C\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2, \\ &\tfrac{a\xi^4}{lk_0\left(1+\xi^2\right)^3}\;\left|\operatorname{Re}\left\langle\, \widehat{\phi},\, \widehat{z}\right\rangle\right| \leq C\left(\varepsilon_4\right)\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2 + \tfrac{\varepsilon_4}{6}\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2, \\ &\tfrac{m}{lk_0}\tfrac{\xi^4}{\left(1+\xi^2\right)^3}\left|\operatorname{Re}\left\langle\, \widehat{\phi},\;\; \widehat{\theta}\right\rangle\right| \leq C\left(\varepsilon_4\right)\xi^2\left|\widehat{\theta}\right|^2 + \tfrac{\varepsilon_4}{6}\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2, \\ &\tfrac{\gamma\left(1-l^2k_0^2\right)\xi^2}{l^{\,2}k_0\left(1+\xi^2\right)^3}\left|\operatorname{Re}\left\langle i\xi\,\widehat{\phi},\;\; \widehat{\eta}\right\rangle\right| \leq C\left(\varepsilon_4\right)\left|\widehat{\eta}\right|^2 + \tfrac{\varepsilon_4}{6}\tfrac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2, \end{aligned}$$

then, we have

$$\begin{aligned}\frac{\xi^4}{\left(1+\xi^2\right)^3}\left|\widehat{u}\right|^2 + \frac{\partial}{\partial t}\mathcal{F}_4\left(\xi,t\right) &\leq C\left(\varepsilon_4\right)\left|\widehat{\eta}\right|^2 + C\left(\varepsilon_4\right)\xi^2\left|\widehat{\theta}\right|^2 + C\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2 \\ &\quad C\left(\varepsilon_4\right)\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2 + C\frac{\xi^4}{\left(1+\xi^2\right)^3}\left|\widehat{v}\right|^2 + \varepsilon_4\frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2, \end{aligned} \tag{3.47}$$

where

$$\begin{aligned}\mathcal{F}_4\left(\xi,t\right) &= \frac{2\xi^2}{\left(1+\xi^2\right)^3}\operatorname{Re}\left\langle \widehat{u}, i\xi\,\widehat{v}\right\rangle + \frac{2l\,k_0\xi^2}{\left(1+\xi^2\right)^3}\operatorname{Re}\left\langle \widehat{y}, i\xi\,\widehat{\phi}\right\rangle \\ &\quad - \frac{2\left[1 - l^2k_0^2\right]\xi^2}{lk_0\left(1+\xi^2\right)^3}\operatorname{Re}\left\langle\, \widehat{\phi}, i\xi\widehat{y}\right\rangle - \frac{2\left[1 - l^2k_0^2\right]\xi^2}{l^{\,2}k_0\left(1+\xi^2\right)^3}\operatorname{Re}\left\langle\, i\xi\widehat{\phi},\, \widehat{\eta}\right\rangle. \end{aligned} \tag{3.48}$$

For the case (3.5), we proceed as follow:

Multiplying (3.42) by $\frac{\xi^2}{(1+\xi^2)^3}$, applying Young's inequality and the following estimates, for any $\varepsilon_4>0$, given by

$$\begin{aligned}
&\frac{\gamma\xi^2}{\alpha_1 l^2 k_0 (1+\xi^2)^3}\left|(1-l^2k_0^2)\operatorname{Re}\left\langle\frac{\widehat{\eta}}{(l^2+\xi^2)}, i\xi\,\widehat{\phi}\right\rangle\right| \le C(\varepsilon_4)\left|\widehat{\eta}\right|^2+\frac{\varepsilon_4}{6}\frac{\xi^4}{(1+\xi^2)^2}\left|\widehat{\phi}\right|^2,\\
&\frac{\xi^4}{\alpha_1 l(1+\xi^2)^3}\left|\operatorname{Re}\langle\widehat{y},\widehat{\eta}\rangle\right| \le C\left|\widehat{\eta}\right|^2+C\frac{\xi^4}{(1+\xi^2)^2}\left|\widehat{y}\right|^2,\\
&\frac{\xi^2}{\alpha_1(1+\xi^2)^3}\left|\frac{\left(l^2\left[1-l^2k_0^2+2l^2\right]+\left(l^2+l^2k_0^2-1\right)\xi^2\right)}{l(l^2+\xi^2)}\operatorname{Re}\langle\widehat{\eta}, i\xi\widehat{u}\rangle\right| \le C\left|\widehat{\eta}\right|^2+\frac{1}{2}\frac{\xi^4}{(1+\xi^2)^3}\left|\widehat{u}\right|^2,\\
&\frac{a}{\alpha_1 l k_0}\frac{\xi^4}{(1+\xi^2)^3}\left|\operatorname{Re}\left\langle\widehat{\phi},\ \widehat{z}\right\rangle\right| \le C(\varepsilon_4)\frac{\xi^4}{(1+\xi^2)^2}\left|\widehat{z}\right|^2+\frac{\varepsilon_4}{6}\frac{\xi^4}{(1+\xi^2)^2}\left|\widehat{\phi}\right|^2,\\
&\frac{m}{\alpha_1 l k_0}\frac{\xi^4}{(1+\xi^2)^3}\left|\operatorname{Re}\left\langle\widehat{\phi},\ \widehat{\theta}\right\rangle\right| \le C(\varepsilon_4)\xi^2\left|\widehat{\theta}\right|^2+\frac{\varepsilon_4}{6}\frac{\xi^4}{(1+\xi^2)^2}\left|\widehat{\phi}\right|^2,
\end{aligned}$$

where $\alpha_1=\min\left(1,\left(1+l^2\left(1-k_0^2\right)\right)\right)$. Then we have

$$\begin{aligned}
\frac{\xi^4}{(1+\xi^2)^3}\left|\widehat{u}\right|^2+\frac{\partial}{\partial t}\mathcal{F}_4(\xi,t) \le\ & C(\varepsilon_4)\left|\widehat{\eta}\right|^2+C(\varepsilon_4)\xi^2\left|\widehat{\theta}\right|^2+C\frac{\xi^4}{(1+\xi^2)^2}\left|\widehat{y}\right|^2\\
&+C(\varepsilon_4)\frac{\xi^4}{(1+\xi^2)^2}\left|\widehat{z}\right|^2+C\frac{\xi^4}{(1+\xi^2)^3}\left|\widehat{v}\right|^2+\varepsilon_4\frac{\xi^4}{(1+\xi^2)^2}\left|\widehat{\phi}\right|^2,
\end{aligned}\tag{3.49}$$

where

$$\begin{aligned}
\mathcal{F}_4(\xi,t) =\ & \frac{2\left[1-l^2k_0^2\right]}{\alpha_1 l\, k_0\,(l^2+\xi^2)}\frac{\xi^4}{(1+\xi^2)^3}\operatorname{Re}\left\langle\widehat{u},\widehat{\phi}\right\rangle\\
&+\frac{2\xi^2}{\alpha_1(1+\xi^2)^3}\operatorname{Re}\langle\widehat{u}, i\xi\,\widehat{v}\rangle+\frac{2\xi^4}{\alpha_1 l k_0(1+\xi^2)^5}\operatorname{Re}\left\langle\widehat{y}, i\xi\,\widehat{\phi}\right\rangle\\
&-\frac{2\left[1-l^2k_0^2\right]\xi^2}{\alpha_1 k_0\,(l^2+\xi^2)(1+\xi^2)^3}\operatorname{Re}\left\langle i\xi\widehat{\phi},\ \widehat{\eta}\right\rangle.
\end{aligned}\tag{3.50}$$

For the cases (3.6 − 3.7), we proceed as follow:

Multiplying (3.35) by $\frac{\xi^4}{(1+\xi^2)^4}$, applying Young's inequality and the following estimates, for any $\varepsilon_4>0$, given by

$$\begin{aligned}
&\frac{\xi^4}{(1+\xi^2)^4}\left|(1-l^2k_0^2)\operatorname{Re}\langle i\xi\widehat{u},\ \widehat{y}\rangle\right| \le C\frac{\xi^4}{(1+\xi^2)^2}\left|\widehat{y}\right|^2+\frac{1}{4}\frac{\xi^6}{(1+\xi^2)^4}\left|\widehat{u}\right|^2,\\
&\frac{l\xi^4}{(1+\xi^2)^4}\operatorname{Re}\langle i\xi\widehat{u},\ \widehat{\eta}\rangle \le C\left|\widehat{\eta}\right|^2+\frac{1}{4}\frac{\xi^6}{(1+\xi^2)^4}\left|\widehat{u}\right|^2,\\
&\frac{l\,k_0^2\,\xi^6}{(1+\xi^2)^4}\left|\operatorname{Re}\langle\widehat{y},\widehat{\eta}\rangle\right| \le C\left|\widehat{\eta}\right|^2+C\frac{\xi^4}{(1+\xi^2)^2}\left|\widehat{y}\right|^2,\\
&\frac{al\,k_0\,\xi^6}{(1+\xi^2)^4}\left|\operatorname{Re}\left\langle\ \widehat{z},\widehat{\phi}\right\rangle\right| \le C(\varepsilon_4)\frac{\xi^4}{(1+\xi^2)^2}\left|\widehat{z}\right|^2+\frac{\varepsilon_4}{4}\frac{\xi^6}{(1+\xi^2)^3}\left|\widehat{\phi}\right|^2,\\
&\frac{l\,k_0 m\,\xi^6}{(1+\xi^2)^4}\left|\operatorname{Re}\left\langle\widehat{\theta},\ \widehat{\phi}\right\rangle\right| \le C(\varepsilon_4)\xi^2\left|\widehat{\theta}\right|^2+\frac{\varepsilon_4}{4}\frac{\xi^6}{(1+\xi^2)^3}\left|\widehat{\phi}\right|^2,
\end{aligned}$$

then, we have

$$
\begin{aligned}
& \frac{\xi^6}{\left(1+\xi^2\right)^4}|\widehat{u}|^2+\frac{\partial}{\partial t} \mathcal{F}_4(\xi, t) \qquad (3.51)\\
\leq\; & C|\widehat{\eta}|^2+C\left(\varepsilon_4\right) \xi^2\left|\widehat{\theta}\right|^2+C\left(\varepsilon_4\right) \frac{\xi^4}{\left(1+\xi^2\right)^2}|\widehat{z}|^2 \\
& +C \frac{\xi^6}{\left(1+\xi^2\right)^4}|\widehat{v}|^2+C \frac{\xi^4}{\left(1+\xi^2\right)^2}|\widehat{y}|^2+\varepsilon_4 \frac{\xi^6}{\left(1+\xi^2\right)^3}\left|\widehat{\phi}\right|^2,
\end{aligned}
$$

where

$$
\mathcal{F}_4(\xi, t)=\frac{2 \xi^4}{\left(1+\xi^2\right)^4} \operatorname{Re}\langle\widehat{u}, i \xi\, \widehat{v}\rangle+\frac{2 l\, k_0 \xi^4}{\left(1+\xi^2\right)^4} \operatorname{Re}\left\langle\widehat{y}, i \xi\, \widehat{\phi}\right\rangle . \qquad (3.52)
$$

Step5. Multiplying $(2.2)_6$ by $i\xi\,\widehat{\phi}$ and using $(2.2)_5$, we have

$$
\begin{aligned}
0 &= \left\langle\widehat{\eta}_t, i \xi\, \widehat{\phi}\right\rangle-k_0\, \xi^2\left|\widehat{\phi}\right|^2-l\left\langle\widehat{v}, i \xi\, \widehat{\phi}\right\rangle+\gamma\left\langle\widehat{\eta}, i \xi\, \widehat{\phi}\right\rangle \\
&= \frac{\partial}{\partial t}\left\langle\widehat{\eta}, i \xi\, \widehat{\phi}\right\rangle+k_0 \xi^2|\widehat{\eta}|^2+l\, k_0\langle\widehat{\eta}, i \xi \widehat{u}\rangle-k_0\, \xi^2\left|\widehat{\phi}\right|^2-l\left\langle\widehat{v}, i \xi\, \widehat{\phi}\right\rangle+\gamma\left\langle\widehat{\eta}, i \xi\, \widehat{\phi}\right\rangle,
\end{aligned}
$$

then

$$
k_0\, \xi^2\left|\widehat{\phi}\right|^2-\frac{\partial}{\partial t}\left\langle\widehat{\eta}, i \xi\, \widehat{\phi}\right\rangle=k_0 \xi^2|\widehat{\eta}|^2+l\, k_0\langle\widehat{\eta}, i \xi \widehat{u}\rangle-l\left\langle\widehat{v}, i \xi\, \widehat{\phi}\right\rangle+\gamma\left\langle\widehat{\eta}, i \xi\, \widehat{\phi}\right\rangle . \qquad (3.53)
$$

Multiplying $(2.2)_2$ by $\widehat{\phi}$ and using $(2.2)_5$, we have

$$
\begin{aligned}
0 &= \frac{\partial}{\partial t}\left\langle\widehat{u}, \widehat{\phi}\right\rangle-\left\langle\widehat{u}, \widehat{\phi}_t\right\rangle-\left\langle i \xi\, \widehat{v}, \widehat{\phi}\right\rangle-l\, k_0\left|\widehat{\phi}\right|^2 \\
&= \frac{\partial}{\partial t}\left\langle\widehat{u}, \widehat{\phi}\right\rangle-k_0\langle\widehat{u},\, i\, \xi\, \widehat{\eta}\rangle+l\, k_0\, |\widehat{u}|^2-\left\langle i \xi\, \widehat{v}, \widehat{\phi}\right\rangle-l\, k_0\left|\widehat{\phi}\right|^2,
\end{aligned}
$$

then, we have

$$
l^{\,2} k_0\left|\widehat{\phi}\right|^2-l \frac{\partial}{\partial t}\left\langle\widehat{u}, \widehat{\phi}\right\rangle=-l k_0\langle\widehat{u},\, i\, \xi\, \widehat{\eta}\rangle+l^2\, k_0\, |\widehat{u}|^2-l\left\langle i \xi\, \widehat{v}, \widehat{\phi}\right\rangle . \qquad (3.54)
$$

Adding (3.53) and (3.54), we get

$$
\begin{aligned}
& \left|\widehat{\phi}\right|^2-\frac{l}{k_0\,\left(l^{\,2}+\xi^2\right)} \frac{\partial}{\partial t} \operatorname{Re}\left\langle\widehat{u}, \widehat{\phi}\right\rangle-\frac{1}{k_0\,\left(l^{\,2}+\xi^2\right)} \frac{\partial}{\partial t}\left(\operatorname{Re}\left\langle\widehat{\eta}, i \xi\, \widehat{\phi}\right\rangle\right) \qquad (3.55)\\
=\; & \frac{\xi^2}{\left(l^{\,2}+\xi^2\right)}|\widehat{\eta}|^2+\frac{l^2}{\left(l^{\,2}+\xi^2\right)}\, |\widehat{u}|^2+\frac{2 l}{\left(l^{\,2}+\xi^2\right)} \operatorname{Re}\langle\widehat{\eta}, i \xi \widehat{u}\rangle \\
& +\frac{\gamma}{k_0\,\left(l^{\,2}+\xi^2\right)} \operatorname{Re}\left\langle\widehat{\eta}, i \xi\, \widehat{\phi}\right\rangle,
\end{aligned}
$$

as

$$
\frac{\gamma}{k_0\,\left(l^{\,2}+\xi^2\right)}\left|\operatorname{Re}\left\langle\widehat{\eta}, i \xi\, \widehat{\phi}\right\rangle\right| \leq C|\widehat{\eta}|^2+\frac{1}{2}\left|\widehat{\phi}\right|^2,
$$

we obtain

$$\left|\widehat{\phi}\right|^2-\frac{2l}{k_0\ \left(l^{\,2}+\xi^2\right)}\frac{\partial}{\partial t}\operatorname{Re}\left\langle \widehat{u},\widehat{\phi}\right\rangle-\frac{2}{k_0\ \left(l^{\,2}+\xi^2\right)}\frac{\partial}{\partial t}\left(\operatorname{Re}\left\langle \widehat{\eta},i\xi\,\widehat{\phi}\right\rangle\right)\leq C\ \left|\widehat{\eta}\right|^2+C\ \left|\widehat{u}\right|^2. \quad (3.56)$$

For the case (3.1)**,** we proceed as follow:

Multiplying (3.56) by $\frac{\xi^2}{\left(1+\xi^2\right)}$,we deduce

$$\frac{\xi^2}{\left(l^{\,2}+\xi^2\right)}\left|\widehat{\phi}\right|^2+\frac{\partial}{\partial t}\mathcal{F}_5\left(\xi,t\right)\leq C\ \left|\widehat{\eta}\right|^2+C\frac{\xi^2}{\left(1+\xi^2\right)}\ \left|\widehat{u}\right|^2, \quad (3.57)$$

where

$$\mathcal{F}_5\left(\xi,t\right)=-\frac{2l\xi^2}{k_0\ \left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)}\operatorname{Re}\left\langle \widehat{u},\widehat{\phi}\right\rangle-\frac{2\xi^2}{k_0\ \left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)}\operatorname{Re}\left\langle \widehat{\eta},i\xi\,\widehat{\phi}\right\rangle. \quad (3.58)$$

For the cases $(3.2-3.3)$**,** we proceed as follow:

Multiplying (3.56) by $\frac{\xi^4}{\left(1+\xi^2\right)^2}$,we deduce

$$\frac{\xi^4}{\left(l^{\,2}+\xi^2\right)^2}\left|\widehat{\phi}\right|^2+\frac{\partial}{\partial t}\mathcal{F}_5\left(\xi,t\right)\leq C\ \left|\widehat{\eta}\right|^2+C\frac{\xi^4}{\left(1+\xi^2\right)^2}\ \left|\widehat{u}\right|^2, \quad (3.59)$$

where

$$\mathcal{F}_5\left(\xi,t\right)=-\frac{2l\xi^4}{k_0\ \left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle \widehat{u},\widehat{\phi}\right\rangle-\frac{2\xi^4}{k_0\ \left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle \widehat{\eta},i\xi\,\widehat{\phi}\right\rangle. \quad (3.60)$$

For the cases $(3.4-3.5)$**,** we proceed as follow:

Multiplying (3.55) by $\frac{\xi^4}{\left(1+\xi^2\right)^2}$, applying Young's inequality and the following estimates, given by

$$\begin{array}{c}\frac{2l\xi^4}{\left(1+\xi^2\right)^2}\left|\operatorname{Re}\left\langle \widehat{\eta},\frac{i\xi}{\left(l^{\,2}+\xi^2\right)}\widehat{u}\right\rangle\right|\leq C\ \left|\widehat{\eta}\right|^2+C\frac{\xi^4}{\left(1+\xi^2\right)^3}\ \left|\widehat{u}\right|^2,\\ \frac{\gamma\xi^4}{k_0\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle \widehat{\eta},\frac{i\xi}{\left(l^{\,2}+\xi^2\right)}\widehat{\phi}\right\rangle\leq C\ \left|\widehat{\eta}\right|^2+\frac{1}{2}\frac{\xi^4}{\left(1+\xi^2\right)^2}\ \left|\widehat{\phi}\right|^2,\end{array}$$

then, we have

$$\frac{\xi^4}{\left(l^{\,2}+\xi^2\right)^2}\left|\widehat{\phi}\right|^2+\frac{\partial}{\partial t}\mathcal{F}_5\left(\xi,t\right)\leq C\ \left|\widehat{\eta}\right|^2+C\frac{\xi^4}{\left(1+\xi^2\right)^3}\ \left|\widehat{u}\right|^2, \quad (3.61)$$

where

$$\mathcal{F}_5\left(\xi,t\right)=-\frac{2l\xi^4}{k_0\ \left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle \widehat{u},\widehat{\phi}\right\rangle-\frac{2\xi^4}{k_0\ \left(l^{\,2}+\xi^2\right)\left(1+\xi^2\right)^2}\operatorname{Re}\left\langle \widehat{\eta},i\xi\,\widehat{\phi}\right\rangle. \quad (3.62)$$

For the cases $(3.6-3.7)$**,** we proceed as follow:

Multiplying (3.55) by $\dfrac{\xi^6}{\left(1+\xi^2\right)^3}$, applying Young's inequality and the following estimates, given by

$$\frac{2l\xi^6}{\left(1+\xi^2\right)^3}\left|\operatorname{Re}\left\langle\widehat{\eta},\frac{i\xi}{\left(l^2+\xi^2\right)}\widehat{u}\right\rangle\right|\leq C\,|\widehat{\eta}|^2+C\frac{\xi^6}{\left(1+\xi^2\right)^4}\,|\widehat{u}|^2,$$
$$\frac{\gamma}{k_0\left(l^2+\xi^2\right)}\frac{\xi^6}{\left(1+\xi^2\right)^3}\left|\operatorname{Re}\left\langle\widehat{\eta},i\xi\,\widehat{\phi}\right\rangle\right|\leq C\,|\widehat{\eta}|^2+\frac{1}{2}\frac{\xi^6}{\left(1+\xi^2\right)^3}\,\left|\widehat{\phi}\right|^2,$$

then, we have

$$\frac{\xi^6}{\left(1+\xi^2\right)^3}\left|\widehat{\phi}\right|^2+\frac{\partial}{\partial t}\mathcal{F}_5(\xi,t)\leq C\,|\widehat{\eta}|^2+C\frac{\xi^6}{\left(1+\xi^2\right)^4}\,|\widehat{u}|^2, \tag{3.63}$$

where

$$\mathcal{F}_5(\xi,t)=-\frac{l\xi^6}{k_0\,\left(l^2+\xi^2\right)\left(1+\xi^2\right)^3}\operatorname{Re}\left\langle\widehat{u},\widehat{\phi}\right\rangle-\frac{\xi^6}{k_0\,\left(l^2+\xi^2\right)\left(1+\xi^2\right)^3}\operatorname{Re}\left\langle\widehat{\eta},i\xi\,\widehat{\phi}\right\rangle. \tag{3.64}$$

Step6. In this step, we make the appropriate combination of the above obtained functionals to build an appropriate Lyapunov functional $\mathcal{L}(\xi,t)$.

We introduce the following Lyapunov functional $\mathcal{L}(\xi,t)$ as follow:

$$\mathcal{L}(\xi,t)=N\,\widehat{E}(\xi,t)+N_1\mathcal{F}_1(\xi,t)+N_2\,\mathcal{F}_2(\xi,t)+N_3\,\mathcal{F}_3(\xi,t)+N_4\,\mathcal{F}_4(\xi,t)+\mathcal{F}_5(\xi,t), \tag{3.65}$$

where N, N_i for $i=1..5$, are positive constants that will be fixed later.

For the case (3.1)**,** taking the derivative of $\mathcal{L}(\xi,t)$ with respect to t and making use of (3.6), (3.16), (3.25), (3.37) and (3.57),we find

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathcal{L}(\xi,t)+\left(N_1-C(\varepsilon_2)N_2-C(\varepsilon_3)N_3-CN_4\right)\frac{\xi^2}{\left(1+\xi^2\right)}|\widehat{y}|^2\\
&+\left(N_2-\varepsilon_1N_1-CN_3\right)\frac{\xi^2}{\left(1+\xi^2\right)}|\widehat{z}|^2+\left(N_3-CN_4\right)\frac{\xi^2}{\left(1+\xi^2\right)}|\widehat{v}|^2\\
&+\left(N_4-\varepsilon_2N_2-\varepsilon_3N_3-C\right)\frac{\xi^2}{\left(1+\xi^2\right)}\,|\widehat{u}|^2\\
&+\left(1-\varepsilon_1N_1-\varepsilon_4N_4\right)\frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{\phi}\right|^2\\
\leq\;&-\left(Nk_1-C(\varepsilon_1)N_1-CN_2-CN_3-C(\varepsilon_4)N_4\right)\xi^2\left|\widehat{\theta}\right|^2\\
&-\left(N\gamma-CN_1-CN_2-CN_3-CN_4-C\right)|\widehat{\eta}|^2.
\end{aligned} \tag{3.66}$$

Now, we fix the constants in (3.66) as follows:

$$\varepsilon_1=\frac{1}{4N_1};\ \varepsilon_4=\frac{1}{4N_4}\ ;\ \varepsilon_2=\frac{1}{4N_2};\ \varepsilon_3=\frac{1}{4N_3}\ \text{and}\ N_4=1+C\ .$$
$$N_3=\frac{1}{2}+CN_4\ ;\ N_2=\frac{1}{2}\ +\varepsilon_1N_1+CN_3\ \text{and}\ N_1=\frac{1}{2}\ +C(\varepsilon_2)N_2+C(\varepsilon_3)N_3+CN_4.$$

Finally, we choose N large enough such that

$$N > \max\left(\frac{1}{k_1}\left[C\left(\varepsilon_1\right)N_1 + CN_2 + CN_3 + C\left(\varepsilon_4\right)N_4\right], \frac{1}{\gamma}\left[CN_1 + CN_2 + CN_3 + CN_4 + C\right]\right).$$

With these choices, (3.66) takes the form

$$\frac{\partial}{\partial t}\mathcal{L}\left(\xi, t\right) + c_1\mathcal{F}\left(\xi, t\right) \leq 0, \tag{3.67}$$

where c_1 is a positive constant, and

$$\begin{aligned}\mathcal{F}\left(\xi, t\right) &= \frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{v}\right|^2 + \frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{u}\right|^2 + \frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{y}\right|^2 \\ &\quad + \frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{z}\right|^2 + \frac{\xi^2}{\left(1+\xi^2\right)}\left|\widehat{\phi}\right|^2 + \xi^2\left|\widehat{\theta}\right|^2 + \left|\widehat{\eta}\right|^2\end{aligned} \tag{3.68}$$

Since N is large enough then there exist two positive constants β_1 and β_2

$$\beta_1\widehat{E}\left(\xi, t\right) \leq \mathcal{L}\left(\xi, t\right) \leq \beta_2\widehat{E}\left(\xi, t\right). \tag{3.69}$$

From (3.68) we deduce that

$$\mathcal{F}\left(\xi, t\right) \geq \lambda_1\left(\xi\right)\ \widehat{E}\left(\xi, t\right), \tag{3.70}$$

where$\lambda_1\left(\xi\right) = \frac{\xi^2}{\left(1+\xi^2\right)}$. Consequently, from(3.67), (3.69) and (3.70), we can find the positive constants C and c such that

$$E\left(\xi, t\right) = \left|\widehat{U}\left(\xi, t\right)\right| \leq C\,\widehat{E}\left(\xi, 0\right)e^{-c\,\lambda_1(\xi)\,t} = C\,\left|\widehat{U}\left(\xi, 0\right)\right|e^{-c\,\lambda_1(\xi)\,t}.$$

For the cases $(3.2 - 3.3)$. Following the same steps as in the previous case, we will have

$$\frac{\partial}{\partial t}\mathcal{L}\left(\xi, t\right) + c_1\mathcal{F}\left(\xi, t\right) \leq 0,$$

where

$$\begin{aligned}\mathcal{F}\left(\xi, t\right) &= \frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{v}\right|^2 + \frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{u}\right|^2 + \frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{y}\right|^2 \\ &\quad + \frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{z}\right|^2 + \frac{\xi^4}{\left(1+\xi^2\right)^2}\left|\widehat{\phi}\right|^2 + \xi^2\left|\widehat{\theta}\right|^2 + \left|\widehat{\eta}\right|^2\end{aligned}$$

and then, we obtain

$$E\left(\xi, t\right) = \left|\widehat{U}\left(\xi, t\right)\right| \leq C\,\widehat{E}\left(\xi, 0\right)e^{-c\,\lambda_2(\xi)\,t} = C\,\left|\widehat{U}\left(\xi, 0\right)\right|e^{-c\,\lambda_2(\xi)\,t},$$

where $\lambda_2\left(\xi\right) = \frac{\xi^4}{\left(1+\xi^2\right)^2}$.

For the cases $(3.4-3.5)$. Following the same computation as before, we obtain:

$$\frac{\partial}{\partial t}\mathcal{L}\left(\xi,t\right)+c_{1}\mathcal{F}\left(\xi,t\right)\leq 0,$$

where

$$\begin{aligned}\mathcal{F}\left(\xi,t\right) &= \frac{\xi^{4}}{\left(1+\xi^{2}\right)^{3}}\left|\widehat{v}\right|^{2}+\frac{\xi^{4}}{\left(1+\xi^{2}\right)^{3}}\left|\widehat{u}\right|^{2}+\frac{\xi^{4}}{\left(1+\xi^{2}\right)^{2}}\left|\widehat{y}\right|^{2}\\ &\quad +\frac{\xi^{4}}{\left(1+\xi^{2}\right)^{2}}\left|\widehat{z}\right|^{2}+\frac{\xi^{4}}{\left(1+\xi^{2}\right)^{2}}\left|\widehat{\phi}\right|^{2}+\xi^{2}\left|\widehat{\theta}\right|^{2}+\left|\widehat{\eta}\right|^{2},\end{aligned}$$

and then, we obtain

$$E\left(\xi,t\right)=\left|\widehat{U}\left(\xi,t\right)\right|\leq C\,\widehat{E}\left(\xi,0\right)e^{-c\,\lambda_{3}(\xi)\,t}=C\,\left|\widehat{U}\left(\xi,0\right)\right|e^{-c\,\lambda_{3}(\xi)\,t},$$

where $\lambda_{3}\left(\xi\right)=\frac{\xi^{4}}{\left(1+\xi^{2}\right)^{3}}$.

For the cases $(3.6-3.7)$. Following the same computation as before, we get:

$$\frac{\partial}{\partial t}\mathcal{L}\left(\xi,t\right)+c_{1}\mathcal{F}\left(\xi,t\right)\leq 0,$$

where

$$\begin{aligned}\mathcal{F}\left(\xi,t\right) &= \frac{\xi^{6}}{\left(1+\xi^{2}\right)^{4}}\left|\widehat{v}\right|^{2}+\frac{\xi^{6}}{\left(1+\xi^{2}\right)^{4}}\left|\widehat{u}\right|^{2}+\frac{\xi^{4}}{\left(1+\xi^{2}\right)^{2}}\left|\widehat{y}\right|^{2}\\ &\quad +\frac{\xi^{4}}{\left(1+\xi^{2}\right)^{2}}\left|\widehat{z}\right|^{2}+\frac{\xi^{6}}{\left(1+\xi^{2}\right)^{3}}\left|\widehat{\phi}\right|^{2}+\xi^{2}\left|\widehat{\theta}\right|^{2}+\left|\widehat{\eta}\right|^{2},\end{aligned}$$

and then, we obtain

$$E\left(\xi,t\right)=\left|\widehat{U}\left(\xi,t\right)\right|\leq C\,\widehat{E}\left(\xi,0\right)e^{-c\,\lambda_{4}(\xi)\,t}=C\,\left|\widehat{U}\left(\xi,0\right)\right|e^{-c\,\lambda_{4}(\xi)\,t},$$

where $\lambda_{4}\left(\xi\right)=\frac{\xi^{6}}{\left(1+\xi^{2}\right)^{4}}$. This complete the proof of Proposition 3.1.

3.2 Decay estimates

In this subsection, we establish the decay estimates of the solution $U(x;t)$ of system (2.1). Our main result reads as follow:

Theorem 3.2. *The solution U of the problem* (2.1) *satisfies the following decay estimates for $t\geq 0$*

a- *If* (3.1) *is satisfied, then we have*

$$\left\|\partial_{x}^{k}U\left(t\right)\right\|_{L^{2}}\leq C\left(1+t\right)^{-\frac{1}{4}-\frac{k}{2}}\left\|U_{0}\right\|_{L^{1}}+Ce^{-ct}\left\|\partial_{x}^{k}U_{0}\right\|_{L^{2}}.$$

b- *If* (3.2) *or* (3.3) *is satisfied, then we have*

$$\left\|\partial_x^k U(t)\right\|_{L^2} \le C(1+t)^{-\frac{1}{8}-\frac{k}{4}} \left\|U_0\right\|_{L^1} + Ce^{-ct}\left\|\partial_x^k U_0\right\|_{L^2}.$$

c- *If* (3.4) *or* (3.5) *is satisfied, then we have*

$$\left\|\partial_x^k U(t)\right\|_{L^2} \le C(1+t)^{-\frac{1}{8}-\frac{k}{4}} \left\|U_0(t)\right\|_{L^1} + C(1+t)^{-\frac{\delta}{2}}\left\|\partial_x^{k+\delta} U_0(t)\right\|_{L^2}.$$

d- *If* (3.6) *or* (3.7) *is satisfied, then we have*

$$\left\|\partial_x^k U(t)\right\|_{L^2} \le C(1+t)^{-\frac{1}{12}-\frac{k}{6}} \left\|U_0\right\|_{L^1} + C(1+t)^{-\frac{\delta}{2}}\left\|\partial_x^{k+\delta} U_0\right\|_{L^2},$$

where C and c are positive constants, and k and δ are two positive integers.

Proof. By using the Fourier transform, the proof of theorem 3.2 is reduced to the analysis of the behavior of the spectral parameter in low- frequency or in the high-frequency regions. The proof is based on the pointwise estimates in proposition 3.1. Applying the Plancherel theorem and making use of the inequality in (3.1), we obtain

$$\begin{aligned} \left\|\partial_x^n U(t)\right\|_2^2 &= \int_{\mathbb{R}} \xi^{2n} \left|U(\xi,t)\right|^2 d\xi \qquad (3.1)\\ &\le C\int_{\mathbb{R}} \xi^{2n} e^{-c\lambda(\xi)t}\left|\widehat{U}(\xi,0)\right|^2 d\xi \\ &\le C\int_{|\xi|\le 1} \xi^{2n} e^{-c\lambda(\xi)t}\left|\widehat{U}(\xi,0)\right|^2 d\xi + C\int_{|\xi|\ge 1} \xi^{2n} e^{-c\lambda(\xi)t}\left|\widehat{U}(\xi,0)\right|^2 d\xi \\ &= I_1 + I_2. \end{aligned}$$

The integral here is split into two parts: the low-frequency part ($|\xi| \le 1$) and the high-frequency part ($|\xi| \ge 1$).

Estimation of I_1 :

For the case (3.1) with $\lambda(\xi) = \lambda_1(\xi) = \frac{\xi^2}{1+\xi^2}$.Here, we see that $\lambda(\xi) \ge \frac{\xi^2}{2}$, so that we have

$$I_1 \le C\int_{|\xi|\le 1} \xi^{2n} e^{-c\xi^2 t}\left|\widehat{U}(\xi,0)\right|^2 d\xi.$$

For $1 \le p \le 2$, we choose p such that $\frac{1}{p}+\frac{1}{q}=1$. Also, we take r such that $\frac{1}{r}+\frac{2}{q}=1$. Then, we see that $\frac{1}{2r}=\frac{1}{p}-\frac{1}{2}$. Applying the Holder inequality and the Hausdorff-Young inequality, we can estimate I_1 as

$$\begin{aligned} I_1 &\le C\left\|\xi^{2n} e^{-c\xi^2 t}\right\|_{L^r(|\xi|\le 1)} \left\|\widehat{U}(\xi,0)\right\|_{L^q}^2 \\ &\le C(1+t)^{-\frac{1}{2r}-n}\left\|U(x,0)\right\|_{L^p}^2 = C(1+t)^{-\left(\frac{1}{p}-\frac{1}{2}\right)-n}\left\|U(x,0)\right\|_{L^p}^2. \end{aligned}$$

For the cases (3.2 − 3.3) with $\lambda(\xi) = \lambda_2(\xi) = \frac{\xi^4}{\left(1+\xi^2\right)^2}$, and **for** the cases (3.4 − 3.5) with $\lambda(\xi) = \lambda_3(\xi) = \frac{\xi^4}{\left(1+\xi^2\right)^3}$. Here, one can see that $\lambda(\xi) \ge \frac{\xi^4}{4}$ or $\lambda(\xi) \ge \frac{\xi^4}{8}$, so that we have

$$
\begin{aligned}
I_1 &\leq C \left\| \xi^{2n} \, e^{-c\,\xi^4 t} \right\|_{L^r(|\xi|\leq 1)} \left\| \widehat{U}\left(\xi,0\right) \right\|_{L^q}^2 \\
&\leq C\left(1+t\right)^{-\frac{1}{4r}-\frac{n}{2}} \left\| U\left(x,0\right) \right\|_{L^p}^2 = C\left(1+t\right)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{n}{2}} \left\| U\left(x,0\right) \right\|_{L^p}^2 .
\end{aligned}
$$

For the cases $(3.6-\ 3.7)$ with $\lambda\left(\xi\right)=\lambda_4\left(\xi\right)\ =\ \frac{\xi^6}{\left(1+\xi^2\right)^4}$.It's easy to see that: $\lambda\left(\xi\right)\geq\frac{\xi^6}{16}$, then we have:

$$
\begin{aligned}
I_1 &\leq C \left\| \xi^{2n} \, e^{-c\,\xi^6 t} \right\|_{L^r(|\xi|\leq 1)} \left\| \widehat{U}\left(\xi,0\right) \right\|_{L^q}^2 \\
&\leq C\left(1+t\right)^{-\frac{1}{6r}-\frac{n}{3}} \left\| U\left(x,0\right) \right\|_{L^p}^2 = C\left(1+t\right)^{-\frac{1}{3}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{n}{3}} \left\| U\left(x,0\right) \right\|_{L^p}^2 .
\end{aligned}
$$

Estimation of I_2 :

For the case (3.1) with $\lambda\left(\xi\right)=\lambda_1\left(\xi\right)\ =\frac{\xi^2}{1+\xi^2}$, and **for the cases** $(3.2-3.3)$ with $\lambda\left(\xi\right)=\lambda_2\left(\xi\right)\ =\ \frac{\xi^4}{\left(1+\xi^2\right)^2}$ In the high frequency region, one can see that $\lambda\left(\xi\right)\geq C$ for $|\xi|\geq 1$. Therefore we can estimate I_2 as

$$
\begin{aligned}
I_2 &\leq Ce^{-c\,t} \int\limits_{|\xi|\geq 1} \xi^{2n} \left| \widehat{U}\left(\xi,0\right) \right|^2 d\xi \\
&\leq Ce^{-c\,t} \left\| \partial_x^n U\left(\xi,0\right) \right\|_{L^2}^2 .
\end{aligned}
$$

For the cases $(3.4-3.5)$ with $\lambda\left(\xi\right)=\lambda_3\left(\xi\right)\ =\ \frac{\xi^4}{\left(1+\xi^2\right)^3}$, or **for** the cases $(3.6-3.7)$ with $\lambda\left(\xi\right)=\lambda_4\left(\xi\right)\ =\ \frac{\xi^6}{\left(1+\xi^2\right)^4}$. It's easy to see that $\lambda\left(\xi\right)\geq C$ for $|\xi|\geq\xi^{-2}$. Therefore we can estimate I_2 as

$$
\begin{aligned}
I_2 &\leq C\sup_{|\xi|\geq 1}\left(|\xi|^{-2\delta} \, e^{-c\,|\xi|^{-2}\,t} \right) \int\limits_{|\xi|\geq 1} \xi^{2(n+\delta)} \left| \widehat{U}\left(\xi,0\right) \right|^2 d\xi \\
&\leq C\left(1+t\right)^{-\delta} \left\| \partial_x^{n+\delta} U\left(\xi,0\right) \right\|_{L^2}^2 .
\end{aligned}
$$

Substituting these estimates into (3.1) gives the estimations in Theorem 3.2.

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