Existence and controllability results for second-order neutral stochastic equations with non-Lipschitz coefficient driven by Rosenblatt process

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Abstract

In this paper we consider a class of second-order impulsive stochastic functional differential equations driven simultaneously by a Rosenblatt process and standard Brownian motion in a Hilbert space. We prove an existence and uniqueness result under non-Lipschitz condition which is weaker than Lipschitz one and we establish some conditions ensuring the controllability for the mild solution by means of the Banach fixed point principle. At the end we provide a practical example in order to illustrate the viability of our result.

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1 Introduction

The concept of Controllability gains more attention in the past decade because of its various applications in the field of applied mathematics. Controllability generally means that with the help of set of admissible controls, it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state. For basic concepts about the controllability, reader may refer ([1], [2]).

The second order differential equations play a vital role in constructing the various mathematical and physical model problems. A useful tool for the study of second-order abstract differential equations is the theory of strongly continuous cosine families of operators. Existence and uniqueness of the solution of second-order nonlinear systems and controllability of these systems in Banach spaces have been investigated extensively by many authors, we refer the reader to ([3], [4], [5]).

Recently, there has been a growing interest on the stochastic functional differential equations driven by fractional Brownian motion (here after, fBm). The reader is referred to the works of ([6], [7], [8]). Also in general, it is Gaussian and the calculus for it is much easier than other processes. However, in concrete situations where the Gaussianity is not plausible for the model, one can employ the Rosenblatt process. The theory of Rosenblatt process has been developed accordingly owing to its nice properties, namely self-similarity, stationarity of the increments, long-range dependence, etc. (see [9], [10], [11]). On the other hand, the Rosenblatt processes can also be an input in models where self-similarity is observed in empirical data which appears to be non-Gaussian, one can refer the papers ([12], [13]). In [14], Kumar et al. established the approximate controllability of certain non-autonomous second-order nonlinear differential problems with finite delay by utilizing Schauder's fixed-point theorem. Sathiyaraj et al. [15] studied the controllability of second-order nonlinear stochastic delay systems driven by the Rosenblatt distributions in finite dimensional spaces. In [16], Lakhel and McKibben investigated the controllability of certain class of non-autonomous impulsive neutral evolution stochastic functional differential equations, with time varying delays, driven by a Rosenblatt process. Sakthivel et al. [17] studied the existence of solutions for functional second-order non autonomous stochastic differential equations driven by Rosenblatt process by using Krasnoselskii–Schaefer-type fixed point theorem.

In this paper, we are interested in the second-order neutral stochastic differential equations driven by Brownian motion (or Wiener process) and an independent Rosenblatt process of the type

$$\begin{cases} d\left(x^{'}(t) - h(t, x(t))\right) = Ax(t)dt + f\left(t, x(t)\right)dt + g(t, x(t)dw(t) + \sigma(t)dZ_H(t), \\ x(0) = x_0, \quad x^{'}(0) = x_{00}, \quad t \in [0, T] \end{cases} \tag{1}$$

where $x(.)$ takes values in the separable Hilbert space X, $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$ on X, Let Q_K be a positive, self adjoint and trace class operator on K and let $\mathcal{L}_2(K, X)$ be the space of all Q_K -Hilbert-Schmidt operators acting between K and X equipped with the Hilbert-Schmidt norm $\|.\|_{\mathcal{L}_2}$. w is a Q_K-Wiener process on Hilbert space K. Let Q be a positive, self adjoint and trace class operator on Y and let $\mathcal{L}_2^0(Y, X)$ be the space of all Q -Hilbert-Schmidt operators acting between Y and X equipped with the Hilbert-Schmidt norm $\|.\|_{\mathcal{L}_2^0}$. Z_H is a Q-Rosenblatt process on a Hilbert space Y, the process w and Z_H are independent and h, f, g and σ are given functions to be specified later. Let $(\Omega, \mathcal{F}_T, P)$ be the complete probability space with the natural filtration $\{\mathcal{F}_t \mid t \in [0, T]\}$ generated by random variables $\{Z_H(s),\ w(s),\ s \in [0, T]\}$, let x_0 and x_{00} are $\mathcal{F}_0$-measurable X-valued random variables independent of w and Z_H.

We define the following classes of functions: let $\mathcal{L}_2(\Omega, \mathcal{F}_T, X)$ is the Hilbert space of all $\mathcal{F}_T$-measurable, square integrable variables with values in X, $\mathcal{L}_2^{\mathcal{F}}([0, T], X)$ is the Hilbert space of all square integrable and $\mathcal{F}_t$-adapted processes with values in X, $C\left([0, T], \mathcal{L}_2(\Omega, \mathcal{F}_T, X)\right)$ is a Banach space of continuous maps satisfying the condition $\sup_{t\in[0,T]} \mathbf{E} \|x(t)\|^2 < \infty$ and Δ_2^T is the closed subspace of $C\left([0, T], \mathcal{L}_2(\Omega, \mathcal{F}_T, X)\right)$ consisting of measurable and $\mathcal{F}_t$-adapted processes $x(t)$, then Δ_2^T is a Banach space with the norm defined by

$$\|x\|_{\Delta_2^T} = \left(\sup_{t\in[0,T]} \mathbf{E} \|x(t)\|^2\right)^{\frac{1}{2}}.$$

Motivated by the above works this paper is concerned to prove the existence and uniqueness of mild solution for system (1) under non-Lipschitz conditions, which is more general that Lipschitz and linear growth see ([18], [19]). Further, controllability problem is discussed for system (1), it should be mentioned that the existence and uniqueness of second-order neutral stochastic differential equations driven by Wiener process and an independent Rosenblatt process under non-Lipschitz conditions has not been investigated yet. The rest of this paper is organized as follows, in section 2, we will introduced some notations, basic concepts, and basic results about Rosenblatt process, Wiener integral with respect to it over Hilbert spaces. In section 3 and 4, we will prove our main result. In Section 5, we give an example to illustrate the efficiency of the obtained result.

2 Preliminaries

2.1 Cosine Family

Now let use recall some facts about cosine families of operators (see [20])

Definition 1 *The strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ is one parameter family $\{C(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(X,X)$ satisfying*

1. *$C(0) = I$,*
2. *$C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed point $x \in X$,*
3. *$C(t+s) + C(t-s) = 2C(t)C(s)$, for all $t,\ s \in \mathbb{R}$*

Definition 2 *The strongly continuous sine family $\{S(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(X,X)$ associated with $\{C(t)\}_{t\in\mathbb{R}}$ is defined by*

$$S(t)x = \int_0^t C(s)xds, \quad t \in \mathbb{R},\ x \in X$$

Definition 3 *The infinitesimal generator $A : X \to X$ of $\{C(t)\}_{t\in\mathbb{R}}$ is given by*

$$Ax = \frac{d^2}{dt^2}C(t)|_{t=0}, \text{ for all } x \in D(A),$$

with

$$D(A) = \left\{x \in X : C(.)x \in C^2(\mathbb{R}, X)\right\}$$

The infinitesimal generator A is a closed and densely defined operator on X.

Proposition 1 *Suppose that A is the infinitesimal generator of cosine family $\{C(t)\}_{t\in\mathbb{R}}$ with corresponding sine family $\{S(t)\}_{t\in\mathbb{R}}$. Then, it holds:*

1. *There exist a constants $M_A \geq 1$ and $\lambda \geq 0$ such that*

$$\|C(t)\| \leq M_A e^{\lambda|t|} \text{ and hence } \|S(t)\| \leq M_A e^{\lambda|t|}$$

2. *For any* $x \in X$ *and all* $0 \leq s \leq r < \infty$,

$$\int_s^r S(t)xdt \in D(A) \text{ and } A\int_s^r S(t)xdt = [C(r) - C(s)]\, x$$

3. *There exist a constants* $\beta \geq 1$ *such that, for all* $0 \leq s \leq r < \infty$

$$\|S(r) - S(s)\| \leq \beta \left| \int_s^r e^{\lambda|\theta|} d\theta \right|$$

Remark 1 *The uniform boundedness principle, with Proposition (1), implies that both* $\{C_t\}_{t\in[0,T]}$ *and* $\{S(t)\}_{t\in[0,T]}$ *are uniformly bounded, i.e., there exist positive constant* $M = M_A e^{\lambda|T|}$ *such that*

$$\|C(t)\| \leq M \text{ and } \|S(t)\| \leq M \tag{2}$$

2.2 Rosenblatt process

Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let $\{Z_H(t),\ t \in [0, T]\}$ the one-dimensional Rosenblatt process with parameter $H \in (\frac{1}{2}, 1)$. By Tudor [21], it is well known that Z_H has the following integral representation:

$$Z_H(t) = d(H)\int_0^t \int_0^t \left[\int_{y_1\vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1)\frac{\partial K^{H'}}{\partial u}(u, y_2)du\right] dB(y_1)dB(y_2), \tag{3}$$

where

$$\begin{cases} B = \{B(t) : t \in [0, T]\} \text{ is a Wiener process, } H' = \frac{H+1}{2}, \\ d(H) = \frac{1}{H+1}\sqrt{\frac{H}{2(2H-1)}} \text{ is a normalizing constant} \end{cases}$$

and $K^H(t, s)$ is the kernel given by

$$\begin{cases} K^H(t, s) = c_H s^{1/2-H} \int_s^t (u - s)^{H-3/2} u^{H-1/2} du, \quad \text{for } t > s \\ K^H(t, s) = 0, \quad \text{for } t \leq s \end{cases}$$

where $c_H = \sqrt{\frac{H(2H-1)}{\mathcal{B}(2-2H, H-1/2)}}$ and $\mathcal{B}(., .)$ denotes the Beta function. The covariance of the Rosenblatt process $\{Z_H(t),\ t \in [0, T]\}$ satisfies

$$R_H(s, t) := E(Z_H(t)Z_H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad \text{for every } s, t \geq 0$$

Let X and Y be two real, separable Hilbert spaces. Let $Q \in \mathcal{L}_2(Y, X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n \geq 0$ $(n = 1, 2...)$ are non-negative real numbers and $\{e_n\}$ $(n = 1, 2...)$ is a complete orthonormal basis in Y. We define the infinite dimensional Q-Rosenblatt process on Y as

$$Z_H(t) = Z_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n z_n(t). \tag{4}$$

where $(z_n)_{n\geq 0}$ is a family of real independent Rosenblatt process. Note that the series (4) is convergent in $L^2(\Omega)$ for every $t \in [0,T]$, since

$$E\left|Z_Q(t)\right|^2 = \sum_{n=1}^{\infty} \lambda_n E\left(z_n(t)\right)^2 = t^{2H} \sum_{n=1}^{\infty} \lambda_n < \infty$$

Note also that Z_Q has covariance function in the sense that

$$E\left\langle Z_Q(t), x\right\rangle \left\langle Z_Q(s), y\right\rangle = R(s,t)\left\langle Q(x), y\right\rangle, \quad \text{for all } x, y \in Y \text{ and } t, s \in [0,T]$$

In order to define Wiener integrals with respect to the Q-Rosenblatt process. Let $\phi(s) : s \in [0,T]$ be a function with values in $\mathcal{L}_2^0(Y,X)$, such that

$$\sum_{n=1}^{\infty} \left\| K^* \phi Q^{1/2} e_n \right\|_{\mathcal{L}_2^0}^2 < \infty.$$

The Wiener integral with respect to Z_Q is defined by

$$\begin{aligned}
\int_0^t \phi(s) dZ_Q(s) &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n dz_n(s) \\
&= \sum_{n=1}^{\infty} \int_0^t \int_0^t K_H^* \left(\phi e_n\right)(y_1, y_2) dB(y_1) dB(y_2) \qquad (5)
\end{aligned}$$

Now, we end this subsection by stating the following fundamental inequality which was proved in [22].

Lemma 1 *If $\phi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\phi(s)\|_{\mathcal{L}_2^0}^2 ds < \infty$, then the above sum in* (5) *is well defined as a X-valued random variable and we have*

$$E\left\| \int_0^t \phi(s) dZ_H(s) \right\|^2 \leq 2Ht^{2H-1} \int_0^t \|\phi(s)\|_{\mathcal{L}_2^0}^2 ds$$

3 Existence and uniqueness of mild solution

In this section, we study the existence and uniqueness of mild solution for (1). To do this, we first present the definition of mild solutions for the system (1).

Definition 4 *A stochastic process $x \in \Delta_2^T$ is a mild solution of* (1) *if it satisfies the following integral equation*

$$\begin{aligned}
x(t) &= C(t)x_0 + S(t)(x_{00} - h(0, x_0)) + \int_0^t C(t-s) h(s, x(s)) ds \\
&\quad + \int_0^t S(t-s) f(s, x(s) ds + \int_0^t S(t-s) g(s, x(s)) dw(s) \\
&\quad + \int_0^t S(t-s) \sigma(s) dZ_H(s), \quad P - a.s.
\end{aligned}$$

We assume the following non-Lipschitz condition:

(H1) A is the infinitesimal generator of the strongly continuous cosine family $\{C(t)\}_{t\geq 0}$ on X.

(H2) The function $\sigma : [0,T] \to \mathcal{L}_2^0(Y,X)$ is bounded, that is : there exists a positive constant L such $\|\sigma(t)\|^2_{\mathcal{L}_2^0} \leq L$, uniformly in $t \in [0,T]$.

(H3) The functions $h, f : [0,T] \times X \to X$, $g : [0,T] \times X \to \mathcal{L}_2$ are measurable and continuous in x for each fixed $t \in [0,T]$ and there exists a function $G : [0,T] \times [0,+\infty) \to [0,+\infty)$, $(t,v) \to G(t,v)$ such that

$$\mathbf{E}\|h(t,x)\|^2 + \mathbf{E}\|f(t,x)\|^2 + \mathbf{E}\|g(t,x)\|^2_{\mathcal{L}_2} \leq G(t, \mathbf{E}\|x\|^2) \tag{6}$$

for all $t \in [0,T]$ and all $x \in \mathcal{L}^2(\Omega, \mathcal{F}_T, X)$.

(H4) $G(t,v)$ is locally integrable in t for each fixed $v \in [0,+\infty)$ and is continuous non-decreasing in v for each fixed $t \in [0,T]$ and for all $\lambda > 0$, $v_0 \geq 0$ the integral equation $v(t) = v_0 + \lambda \int_0^t G(s, v(s))ds$ has a global solution on $[0,T]$.

(H5) There exists a function $K : [0,T] \times [0,+\infty) \to [0,+\infty)$ such that

$$\begin{array}{l} \mathbf{E}\|h(t,x) - h(t,y)\|^2 + \mathbf{E}\|f(t,x) - f(t,y)\|^2 \leq K(t, \mathbf{E}\|x-y\|^2) \\ \mathbf{E}\|g(t,x) - g(t,y)\|^2_{\mathcal{L}_2} \leq K(t, \mathbf{E}\|x-y\|^2) \end{array} \tag{7}$$

for all $t \in [0,T]$ and all $x, y \in \mathcal{L}^2(\Omega, \mathcal{F}_T, X)$,

(H6) $K(t,v)$ is locally integrable in t for each fixed $v \in [0,+\infty)$ and continuous non-decreasing in v for each fixed $t \in [0,T]$. Moreover, $K(t,0) = 0$ and If a non-negative continuous function $z(t)$, $t \in [0,T]$ satisfies

$$\begin{cases} z(t) \leq \sigma \int_0^t K(s, z(s))ds, & t \in [0,T] \\ z(0) = 0 \end{cases} \tag{8}$$

for some $\sigma > 0$, then $z(t) = 0$ for all $t \in [0,T]$.

Remark 2 *1. If the function K is concave with respect to the second variable for each fixed $t \geq 0$ and*

$$\|h(t,x) - h(t,y)\|^2 + \|f(t,x) - f(t,y)\|^2 + \|g(t,x) - g(t,y)\|^2_{L_2^0} \leq K(t, \|x-y\|^2),$$

for.all $x, y \in X$ and $t \geq 0$. By Jensen's inequality, (7) is satisfied.

2. Let $K(t,v) = \eta(t)\vartheta(v)$, $t \geq 0$, $v \geq 0$ where $\eta(t) \geq 0$ is locally integrable and $\vartheta : [0,+\infty) \to [0,+\infty)$ is a continuous, monotone non-decreasing and concave function with $\vartheta(0) = 0$, $\vartheta(v) > 0$ for $v > 0$ and $\int_{0+} 1/\vartheta(u)du = \infty$. Then the function $K(t,v)$ satisfies assumption (H6).

Now let us give some concrete examples of the function ϑ, let $\epsilon > 0$ be sufficiently small, define

$$\begin{aligned}\vartheta_1(u) &= \begin{cases} u\log(u^{-1}), & 0 \le u \le \epsilon \\ \epsilon\log(\epsilon^{-1}) + \vartheta_1'(\epsilon_-)(u-\epsilon), & u > \epsilon \end{cases} \\ \vartheta_2(u) &= \begin{cases} u\log(u^{-1})\log\log(u^{-1}), & 0 \le u \le \epsilon \\ \epsilon\log(\epsilon^{-1})\log\log(\epsilon^{-1}) + \vartheta_2'(\epsilon_-)(u-\epsilon), & u > \epsilon \end{cases}\end{aligned} \tag{9}$$

Theorem 1 *Assume that the conditions (H1)-(H6) hold, then there exists a unique solution of (1) in Δ_2^T.*

The proof of this theorem is based on the Picard type approximate method. Let us construct a sequence of stochastic process $\{x_n\}_{n\ge 0}$ as follows:

$$\begin{cases} x_0(t) = C(t)x_0 + S(t)(x_{00} - h(0,x_0)) \\ x_{n+1}(t) = C(t)x_0 + S(t)(x_{00} - h(0,x_0)) + \int_0^t C(t-s)h(s,x_n(s))ds \\ + \int_0^t S(t-s)f(s,x_n(s))ds + \int_0^t S(t-s)g(s,x_n(s))dw(s) \\ + \int_0^t S(t-s)\sigma(s)dZ_H(s) \end{cases}$$

Lemma 2 *Under the conditions (H1)-(H5) the sequence $\{x_n\}_{n\ge 0}$ is uniformly bounded in Δ_2^T, i.e., $\mathbf{E}\left[\sup_{s\in[0,T]}\|x_n(s)\|^2\right] \le C$, where C is a constant.*

Proof We have

$$\begin{aligned}\mathbf{E}\|x_{n+1}(t)\|^2 &\le 6\mathbf{E}\|C(t)x_0\|^2 + 6\mathbf{E}\|S(t)(x_{00}-h(0,x_0))\|^2 \\ &\quad +6\mathbf{E}\left\|\int_0^t C(t-s)h(s,x_n(s))ds\right\|^2 + 6\mathbf{E}\left\|\int_0^t S(t-s)f(s,x_n(s))ds\right\|^2 \\ &\quad +6\mathbf{E}\left\|\int_0^t S(t-s)g(s,x_n(s))dw(s)\right\|^2 + 6\mathbf{E}\left\|\int_0^t S(t-s)\sigma(s)dZ_H(s)\right\|^2\end{aligned}$$

By property (2), Holder inequality and Ito isometry theorem, we have

$$\begin{aligned}\mathbf{E}\|x_{n+1}(t)\|^2 &\le 6M^2\mathbf{E}\|x_0\|^2 + 12M^2(\mathbf{E}\|x_{00}\|^2 + \mathbf{E}\|h(0,x_0)\|^2) \\ &\quad +6M^2T\mathbf{E}\int_0^t \left(\mathbf{E}\|h(s,x_n(s))\|^2 + \mathbf{E}\|f(s,x_n(s))\|^2\right)ds \\ &\quad +6M^2\int_0^t \mathbf{E}\|g(s,x_n(s))\|^2_{\mathcal{L}_2}ds + 12M^2HT^{2H-1}\int_0^t \mathbf{E}\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\end{aligned}$$

Then, from (H1)-(H5), we obtain

$$\begin{aligned}\mathbf{E}\|x_{n+1}(t)\|^2 &\le 6M^2\mathbf{E}\|x_0\|^2 + 12M^2\mathbf{E}\|x_{00}\|^2 + 12M^2G\left(0,\mathbf{E}\|x_0\|^2\right) \\ &\quad +6M^2T^2(C_h+C_f)\int_0^t G\left(s,\mathbf{E}\|x_n(s)\|^2\right)ds + 6M^2TC_g\int_0^t G\left(s,\mathbf{E}\|x_n(s)\|^2\right)ds \\ &\quad +12M^2HT^{2H-1}TL \\ &\le C_1 + C_2\int_0^t G\left(s,\mathbf{E}\|x_n(s)\|^2\right)ds\end{aligned}$$

Hence, we get

$$\mathbf{E}\left[\sup_{s\in[0,t]}\|x_{n+1}(s)\|^2\right] \leq C_1 + C_2\int_0^t G\left(s, \mathbf{E}\left[\sup_{r\in[0,s]}\|x_n(r)\|^2\right]\right)ds \tag{10}$$

where

$$\begin{cases} C_1 = 6M^2\left(\mathbf{E}\|x_0\|^2 + 2\mathbf{E}\|x_{00}\|^2 + 2G\left(0, \mathbf{E}\|x_0\|^2\right) + 2HT^{2H-1}TL\right) \\ C_2 = 6M^2T\left(T\left(C_h + C_f\right) + C_g\right) \end{cases}$$

Therefore, from (H4) and inequality (10), there is a $v(t),\ t\in[0,T]$ satisfying

$$v(t) = C_1 + C_2\int_0^t G(s, v(s))ds$$

We shall show, by induction, for $n = 0, 1, 2, ...$

$$\mathbf{E}\left[\sup_{s\in[0,t]}\|x_n(s)\|^2\right] \leq v(t), \quad \forall t\in[0,T] \tag{11}$$

By using the induction argument

$$\begin{aligned}\mathbf{E}\left[\sup_{s\in[0,t]}\|x_0(s)\|^2\right] &= \mathbf{E}\left[\sup_{s\in[0,t]}\|C(s)x_0 + S(s)(x_{00} - h(0,x_0))\|^2\right] \\ &\leq M^2\left(\mathbf{E}\|x_0\|^2 + 2\mathbf{E}\|x_{00}\|^2 + 2G\left(0, \mathbf{E}\|x_0\|^2\right)\right) \leq C_1 \leq v(t), \quad \forall t\in[0,T].\end{aligned}$$

Let use assume that (11) is true for some $n\in\mathbb{N}$, then by (10), the assumption of the mathematical induction and the non-decreasing property of G in v, we have

$$v(t) - \mathbf{E}\left[\sup_{s\in[0,t]}\|x_n(s)\|^2\right] \geq C_2\int_0^t\left(G\left(s, v(s)\right) - G\left(s, \mathbf{E}\left[\sup_{r\in[0,s]}\|x_n(r)\|^2\right]\right)\right)ds \geq 0, \quad \forall t\in[0,T]$$

By induction, we obtain for any $n\in\mathbb{N}$,

$$\mathbf{E}\left[\sup_{s\in[0,t]}\|x_n(s)\|^2\right] \leq v(t) \leq v(T) < \infty$$

Proof [Theorem 1] **Step 1: Existence:** By an argument similar to that in Lemma 2, we have

$$\begin{aligned}
\mathbf{E}\left\|x_{n+m}(t)-x_n(t)\right\|^2 &\leq 3\mathbf{E}\left\|\int_0^t C(t-s)\left(h(s,x_{n+m-1}(s))-h(s,x_{n-1}(s))\right)ds\right\|^2 \\
&\quad +3\mathbf{E}\left\|\int_0^t S(t-s)\left(f(s,x_{n+m-1}(s))-f(s,x_{n-1}(s))\right)ds\right\|^2 \\
&\quad +3\mathbf{E}\left\|\int_0^t S(t-s)\left(f(s,x_{n+m-1}(s))-f(s,x_{n-1}(s))\right)dw(s)\right\|^2 \\
&\leq 2TMC_h\int_0^t \mathbf{E}\left\|h(s,x_{n+m-1}(s))-h(s,x_{n-1}(s))\right\|^2 ds \\
&\quad +2TMC_f\int_0^t \mathbf{E}\left\|f(s,x_{n+m-1}(s))-f(s,x_{n-1}(s))\right\|^2 ds \\
&\quad +2M\int_0^t \mathbf{E}\left\|g(s,x_{n+m-1}(s))-g(s,x_{n-1}(s))\right\|^2 ds
\end{aligned}$$

Thus, we obtain

$$\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_{n+m}(s)-x_n(s)\right\|^2\right]\leq C_3\int_0^t K\left(s,\mathbf{E}\left[\sup_{r\in[0,s]}\left\|x_{n+m-1}(s)-x_{n-1}(s)\right\|^2\right]\right)ds, \qquad (12)$$

where $C_3=2M\left(T(C_h+C_f)+C_g\right)$.
It follows from Lemma (2) that $\sup_{n,m}\left\|x_{n+m-1}-x_{n-1}\right\|^2<\infty$. Therefore, we can apply Fatou's Lemma to (12),

$$\lim\sup_{n,m\to\infty}\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_{n+m}(s)-x_n(s)\right\|^2\right]\leq C_3\int_0^t K\left(s,\lim\sup_{n,m\to\infty}\mathbf{E}\left[\sup_{r\in[0,s]}\left\|x_{n+m-1}-x_{n-1}\right\|^2\right]\right)ds, \qquad (13)$$

Set

$$z(t):=\lim\sup_{n,m\to\infty}\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_{n+m}(s)-x_n(s)\right\|^2\right]$$

Then the above inequality (13) can be rewritten as

$$z(t)\leq C_3\int_0^t K\left(s,z(s)\right)ds,$$

It is obvious that the positive functions $z(t)$ is monotone non-decreasing.on $[0,T]$ with $z(0)=0$ .Hence, from hypothesis (H6) and Barbu (Lemme2.2 [23]) we obtain

$$z(t)=\lim\sup_{n,m\to\infty}\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_{n+m}(s)-x_n(s)\right\|^2\right]=0,\quad \text{for all } t\in[0,T]$$

Thus we get

$$\lim_{n,m\to\infty}\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_{n+m}(s)-x_n(s)\right\|^2\right]=0,\quad \text{for all } t\in[0,T]$$

which implies that the sequence Since $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in Banach space Δ_2^T.
Step 2 : Uniqueness. Assume that x_1 and $x_2 \in \Delta_2^T$ are mild solutions of (1). Analogously as in the proof of (12), we obtain for any $t \in [0,T]$

$$\mathbf{E}\left[\sup_{s\in[0,t]} \|x_1(s) - x_2(s)\|^2\right] \leq C_3 \int_0^t K\left(s, \mathbf{E}\left[\sup_{r\in[0,s]} \|x_1(s) - x_2(s)\|^2\right]\right) ds, \tag{14}$$

Due to hypothesis (H6) and Barbu (Lemme2.2 [23]) we get that $\mathbf{E}\left[\sup_{s\in[0,T]} \|x_1(s) - x_2(s)\|^2\right] = 0$, *i.e.*, $x_1 = x_2$.

4 Controllability result

In this section we state and prove the controllability for second-order neutral stochastic equation driven by Brownian motion and an independent Rosenblatt process of the form

$$\begin{cases} d\left(x^{'}(t) - h(t, x(t))\right) = Ax(t)dt + Bu(t)dt + f\left(t, x(t)\right)dt + g(t, x(t)dw(t) + \sigma(t)dZ_H(t), \\ x(0) = x_0, \quad x^{'}(0) = x_{00}, \quad t \in [0,T] \end{cases} \tag{15}$$

where $h,\ f,\ g,\ \sigma,\ A$ are the same as in the Eq.(1), $B : U \to X$ is a given mapping and the control function u takes values in $U_{ad} = L^2([0,T], U)$, the Hilbert space of admissible control functions for separable Hilbert space U.

Definition 5 *A stochastic process $x \in \Delta_2^T$ is a mild solution of* (15) *if for each $u \in U_{ad}$ it satisfies the following integral equation*

$$\begin{aligned} x(t) \quad &= \quad C(t)x_0 + S(t)(x_{00} - h(0, x_0)) + \int_0^t C(t-s)h(s, x(s))ds \\ &\quad + \int_0^t S(t-s)\left(Bu(s) + f(s, x(s)\right)ds + \int_0^t S(t-s)g(s, x(s))dw(s) \\ &\quad + \int_0^t S(t-s)\sigma(s)dZ_H(s), \quad P - a.s. \end{aligned}$$

Definition 6 *The system* (15) *is said to be controllable on the interval* $[0,T]$, *if for every initial function $x(0) = x_0,\ x^{'}(0) = x_{00}$ and desired final state $x_1 \in X$, there exists a stochastic control $u \in U_{ad}$ such that the mild solution of the system* (15) *corresponding to this control satisfies $x(T) = x_1$.*

The following are the additional assumptions in this section.

(H7) The function $h, f : [0,T] \times X \to X$ and $g : [0,T] \times X \to \mathcal{L}_2(K, X)$ satisfy the linear growth and Lipschitz conditions, that there exist positive constants $C_h,\ C_f$ and C_g such that for $x, y \in X$ and $t \in [0,T]$

$$\begin{array}{lcl} \|h(t,x)\|^2 \leq C_h(1 + \|x\|^2) & & \|h(t,x) - h(t,y)\|^2 \leq C_h \|x - y\|^2 \\ \|f(t,x)\|^2 \leq C_f(1 + \|x\|^2) & , & \|f(t,x) - f(t,y)\|^2 \leq C_f \|x - y\|^2 \\ \|g(t,x)\|_{\mathcal{L}_2}^2 \leq C_g(1 + \|x\|^2) & & \|g(t,x) - g(t,y)\|_{\mathcal{L}_2}^2 \leq C_g \|x - y\|^2 \end{array}$$

(H8) The linear operator $\Gamma : L^2([0,T],U) \to \mathcal{L}_2(\Omega,\mathcal{F}_T,X)$, is defined by

$$\Gamma u = \int_0^T S(T-s)Bu(s)ds$$

has a bounded invertible operator Γ^{-1} which takes values in $L^2([0,T],U)/\ker\Gamma$ and there exist positive constants $M_B,\ M_\Gamma$ such that $\|B\|^2 \le M_B$ and $\left\|\Gamma^{-1}\right\|^2 \le M_\Gamma$.

Now, we describe the controllability result as follows and give its proof.

Theorem 2 *Under assumptions (H1)-(H2) and (H7)-(H8), the system* (15) *is controllable on* $[0,T]$.

Proof Using the hypothesis (H8), for an arbitrary $x_T \in \mathcal{L}_2(\Omega,\mathcal{F}_T,X)$, we define the stochastic control

$$\begin{aligned} u_x(t) \;&=\; \Gamma^{-1}\Big\{x_T - C(T)x_0 - S(T)(x_{00} - h(0,x_0)) - \int_0^T C(T-s)h(s,x(s))ds \\ &\quad - \int_0^T S(T-s)f(s,x(s))ds - \int_0^T S(T-s)g(s,x(s))dw(s) \\ &\quad - \int_0^T S(T-s)\sigma(s)dZ_H(s)\Big\}(t) \end{aligned} \tag{16}$$

Define the operator $\Psi : \Delta_2^T \to \Delta_2^T$ by

$$\begin{aligned} (\Psi x)(t) = C(t)x_0 + S(t)(x_{00} - h(0,x_0)) + \int_0^t C(t-s)h(s,x(s))ds \\ \int_0^t S(t-s)\left(Bu_x(s) + f(s,x(s))\right)ds + \int_0^t S(t-s)g(s,x(s))dw(s) \\ + \int_0^t S(t-s)\sigma(s)dZ_H(s) \end{aligned} \tag{17}$$

Now, we show that the operator Ψ has a fixed point in Δ_2^T which is a mild solution of the system (15). Substituting (16) in (17) we find that $(\Psi x)(T) = x_T$, indicating that the control u_x steers the system from x_0 to x_T in finite time T, which further implies that the system (15) is controllable. We divide the proof into three steps.
Step 1: For any $x \in \Delta_2^T$, $(\Psi x)(t)$ is continuous on the interval $[0,T]$ in L^2-sense

Let $0 \leq t_1 \leq t_2 \leq T$. Then for any fixed $x \in \Delta_2^T$

$$
\begin{aligned}
\mathbf{E}\left\|(\Psi x)(t_2)-(\Psi x)(t_1)\right\|^2 \leq\ & 5\mathbf{E}\left\|\left(C(t_2)-C(t_1)\right)x_0+\left(S(t_2)-S(t_1)\right)\left(x_{00}-h(0,x_0)\right)\right\|^2 \\
& +5\mathbf{E}\left\|\int_0^{t_2}\left[C(t_2-s)h(s,x(s))+S(t_2-s)f(s,x(s))\right]ds\right. \\
& \left.-\int_0^{t_1}\left[C(t_1-s)h(s,x(s))+S(t_1-s)f(s,x(s))\right]ds\right\|^2 \\
& +5\mathbf{E}\left\|\int_0^{t_2}S(t_2-s)f(s,x(s))dw(s)-\int_0^{t_1}S(t_1-s)g(s,x(s))dw(s)\right\|^2 \\
& +5\mathbf{E}\left\|\int_0^{t_2}S(t_2-s)\sigma(s)dZ_H(s)-\int_0^{t_1}S(t_1-s)\sigma(s)dZ_H(s)\right\|^2 \\
& +5\mathbf{E}\left\|\int_0^{t_2}S(t_2-s)Bu_x(s)ds-\int_0^{t_1}S(t_1-s)Bu_x(s)ds\right\|^2 \\
=\ & 5\sum_{1\leq i=5}\mathbf{E}\left\|D_i\right\|^2
\end{aligned}
$$

By the strong continuity of $C(t)$ and $S(t)$, we have

$$\lim_{t_2-t_1\to 0}\left(C(t_2)-C(t_1)\right)x_0+\left(S(t_2)-S(t_1)\right)\left(x_{00}-h(0,x_0)\right)=0$$

From property (2), we have

$$\left\|\left(C(t_2)-C(t_1)\right)x_0+\left(S(t_2)-S(t_1)\right)\left(x_{00}-h(0,x_0)\right)\right\| \leq \mathbf{2}M\left\|x_0\right\|+\mathbf{2}M\left\|x_{00}-h(0,x_0)\right\|$$

Thus we conclude by the Lebesgue's dominated convergence theorem that

$$\lim_{t_2-t_1\to 0}\mathbf{E}\left\|D_1\right\|^2=0$$

For the second term D_2, we have

$$
\begin{aligned}
\left\|D_2\right\| \leq\ & \left\|\int_0^{t_1}\left[\left(C(t_2-s)-C(t_1-s)\right)h(s,x(s))+\left(S(t_2-s)-S(t_1-s)\right)f(s,x(s))\right]ds\right\| \\
& +\left\|\int_{t_1}^{t_2}\left[C(t_2-s)h(s,x(s))+S(t_2-s)f(s,x(s))\right]ds\right\| \\
\leq\ & D_{21}+D_{22}
\end{aligned}
$$

By the Holder inequality

$$\mathbf{E}\left\|D_{21}\right\|^2 \leq t_1\mathbf{E}\int_0^{t_1}\left\|\left(C(t_2-s)-C(t_1-s)\right)h(s,x(s))+\left(S(t_2-s)-S(t_1-s)\right)f(s,x(s))\right\|^2 ds$$

By the strong continuity of $C(t)$ and $S(t)$, we have

$$\lim_{t_2-t_1\to 0}\left(C(t_2-s)-C(t_1-s)\right)h(s,x(s))+\left(S(t_2-s)-S(t_1-s)\right)f(s,x(s))=0$$

By using property (2) and conditions (H7), we obtain

$$\begin{aligned} & \left\| \left(C(t_2 - s) - C(t_1 - s)\right) h(s, x(s)) + \left(S(t_2 - s) - S(t_1 - s)\right) f(s, x(s)) \right\| \\ \leq \; & 2M \left(\|h(s, x(s))\| + \|f(s, x(s))\| \right) \end{aligned}$$

Then we conclude by the Lebesgue's dominated convergence theorem that

$$\lim_{t_2 - t_1 \to 0} \mathbf{E} \|D_{21}\|^2 = 0$$

By property (2), condition (H7) and the Holder inequality, we get

$$\begin{aligned} \mathbf{E} \|D_{22}\|^2 &\leq 2M^2 (t_2 - t_1) \int_{t_1}^{t_2} \mathbf{E} \left(\|h(s, x(s))\|^2 + \|f(s, x(s))\|^2 \right) ds \\ &\leq 2M^2 (C_h + C_f)(t_2 - t_1) \int_0^T \left(\mathbf{E} \|x(s)\|^2 + 1 \right) ds \end{aligned}$$

Thus,

$$\lim_{t_2 - t_1 \to 0} \mathbf{E} \|D_{22}\|^2 = 0$$

Now, for the term D_3, we have

$$\begin{aligned} \|D_3\| &\leq \left\| \int_0^{t_1} \left(S(t_2 - s) - S(t_1 - s)\right) g(s, x(s)) dw(s) \right\| + \left\| \int_{t_1}^{t_2} S(t_2 - s) g(s, x(s)) dw(s) \right\| \\ &\leq \|D_{31}\| + \|D_{32}\| \end{aligned}$$

By Ito isometry theorem, we have

$$\begin{aligned} \|D_{31}\| &\leq \left\| \int_0^{t_1} \left(S(t_2 - s) - S(t_1 - s)\right) g(s, x(s)) dw(s) \right\|^2 \\ &\leq \int_0^{t_1} \left\| \left(S(t_2 - s) - S(t_1 - s)\right) g(s, x(s)) \right\|_{\mathcal{L}_2}^2 ds \end{aligned}$$

Since, by the strong continuity of $S(t)$, we have

$$\lim_{t_2 - t_1 \to 0} \left\| \left(S(t_2 - s) - S(t_1 - s)\right) g(s, x(s)) \right\|_{\mathcal{L}_2}^2 = 0$$

Moreover

$$\left\| \left(S(t_2 - s) - S(t_1 - s)\right) g(s, x(s)) \right\|_{\mathcal{L}_2}^2 \leq 4M^2 \|g(s, x(s))\|_{\mathcal{L}_2}^2$$

Then we conclude by the Lebesgue's dominated convergence theorem that

$$\lim_{t_2 - t_1 \to 0} \mathbf{E} \|D_{31}\|^2 = 0$$

For the second term D_{32}, similarly one get

$$\mathbf{E} \|D_{32}\|^2 \leq 2M^2 \int_{t_1}^{t_2} \|g(s, x(s))\|_{\mathcal{L}_2}^2 ds$$

thus,

$$\lim_{t_2-t_1\to 0}\mathbf{E}\left\|D_{32}\right\|^2=0$$

For D_4, it is obvious that

$$\begin{aligned}\|D_4\| &\le \left\|\int_0^{t_1}\left(S(t_2-s)-S(t_1-s)\right)\sigma(s)dZ_H(s)\right\|+\left\|\int_{t_1}^{t_2}S(t_2-s)\sigma(s)dZ_H(s)\right\| \\ &\le \|D_{41}\|+\|D_{42}\|\end{aligned}$$

By Lemma (1), we have

$$\begin{aligned}\|D_{41}\|^2 &\le \left\|\int_0^{t_1}\left(S(t_2-s)-S(t_1-s)\right)\sigma(s)dZ_H(s)\right\|^2 \\ &\le 2Ht_1^{2H-1}\int_0^{t_1}\left\|\left(S(t_2-s)-S(t_1-s)\right)\sigma(s)\right\|^2_{\mathcal{L}_2^0}ds\end{aligned}$$

We have by the strong continuity of $S(t)$

$$\lim_{t_2-t_1\to 0}\left\|\left(S(t_2-s)-S(t_1-s)\right)\sigma(s)\right\|^2_{\mathcal{L}_2^0}=0$$

Moreover

$$\left\|\left(S(t_2-s)-S(t_1-s)\right)\sigma(s)\right\|^2_{\mathcal{L}_2^0}\le 4M^2\left\|\sigma(s)\right\|^2_{\mathcal{L}_2^0}$$

According to the Lebesgue's dominated convergence theorem, we can obtain

$$\lim_{t_2-t_1\to 0}\mathbf{E}\left\|D_{41}\right\|^2=0$$

In a similar way, we obtain

$$\mathbf{E}\left\|D_{42}\right\|^2\le 4M^2H\left(t_2^{2H-1}-t_1^{2H-1}\right)\int_{t_1}^{t_2}\left\|\sigma(s)\right\|^2_{\mathcal{L}_2^0}ds$$

Thus,

$$\lim_{t_2-t_1\to 0}\mathbf{E}\left\|D_{42}\right\|^2=0$$

Using the Holder inequality, property (2), (H2), (H7) and (H8), we obtain

$$\begin{aligned}\mathbf{E}\left\|u_x(t)\right\|^2 &\le 5M_\Gamma\Big\{\mathbf{E}\left\|x_T\right\|^2+M^2\mathbf{E}\left\|x_0^2\right\|+2M^2(\mathbf{E}\left\|x_{00}\right\|^2+\mathbf{E}\left\|h(0,x_0)\right\|^2) \\ &\quad +M^2\left(T(C_h+C_f)+C_g\right)(1+\|x\|^2_{\Delta_2^T})+2M^2HT^{2H-1}L_\sigma\Big\} \\ &\le M_u(1+\|x\|^2_{\Delta_2^T})\end{aligned} \tag{18}$$

Next, observe that

$$\begin{aligned}\mathbf{E}\left\|D_5\right\|^2 &\le 2\mathbf{E}\left\|\int_0^{t_1}\left(S(t_2-s)-S(t_1-s)\right)Bu_x(s)ds\right\|^2+2\mathbf{E}\left\|\int_{t_1}^{t_2}S(t_2-s)Bu_x(s)ds\right\|^2 \\ &\le 2\left(\mathbf{E}\left\|D_{51}\right\|^2+\mathbf{E}\left\|D_{52}\right\|^2\right)\end{aligned}$$

Use the similar procedure as before, we obtain

$$\mathbf{E}\left\|D_{51}\right\|^2 \leq t_1 \int_0^{t_1} \mathbf{E}\left\|\left(S(t_2-s)-S(t_1-s)\right) B u_x(s)\right\|^2 ds$$

Combing this with the strong continuity of $S(t)$ and inequality (18), we obtain

$$\lim_{t_2-t_1\to 0} \mathbf{E}\left\|D_{51}\right\|^2 = 0$$

For the second term D_{42} similarly one get

$$\mathbf{E}\left\|D_{52}\right\|^2 \leq M^2\left\|B\right\|^2 (t_2-t_1) \int_{t_1}^{t_2} \mathbf{E}\left\|u_x(s)\right\|^2 ds$$

We obtain

$$\lim_{t_2-t_1\to 0} \mathbf{E}\left\|D_{52}\right\|^2 = 0$$

The above argument show that $\lim_{t_2-t_1\to 0} \mathbf{E}\left\|(\Psi x)(t_2)-(\Psi x)(t_1)\right\|^2 = 0$ Thus we conclude $(\Psi x)(t)$ is continuous from the right in $[0,T)$. A similar reasoning show that it is also continuous from the left in $(0,T]$.

Step 2 : The operator Ψ sends Δ_2^T into itself.

Let $x \in \Delta_2^T$, then we have

$$\begin{aligned}
\mathbf{E}\left\|(\Psi x)(t)\right\|^2 \;\leq\;& 7\mathbf{E}\left\|C(t)x_0\right\|^2 + 7\mathbf{E}\left\|S(t)(x_{00}-h(0,x_0))\right\|^2 \\
&+7\mathbf{E}\left\|\int_0^t C(t-s)h(s,x(s))ds\right\|^2 + 7\mathbf{E}\left\|\int_0^t S(t-s)f(s,x(s))ds\right\|^2 \\
&+7\mathbf{E}\left\|\int_0^t S(t-s)g(s,x(s))dw(s)\right\|^2 + 7\mathbf{E}\left\|\int_0^t S(t-s)\sigma(s)dZ_H(s)\right\|^2 \\
&+7\mathbf{E}\left\|\int_0^t S(t-s)Bu_x(s)ds\right\|^2
\end{aligned}$$

By Holder inequality, Ito isometry theorem and property (2), we have

$$\begin{aligned}
\mathbf{E}\left\|(\Psi x)(t)\right\|^2 \;\leq\;& 7M^2\mathbf{E}\left\|x_0\right\|^2 + 14M^2(\mathbf{E}\left\|x_{00}\right\|^2 + \mathbf{E}\left\|h(0,x_0)\right\|^2) \\
&+7M^2T\mathbf{E}\int_0^t \left(\mathbf{E}\left\|h(s,x(s))\right\|^2 + \mathbf{E}\left\|f(s,x(s))\right\|^2\right) ds \\
&+7M^2\int_0^t \mathbf{E}\left\|g(s,x(s))\right\|^2_{\mathcal{L}_2} ds + 14M^2HT^{2H-1}\int_0^t \mathbf{E}\left\|\sigma(s)\right\|^2_{\mathcal{L}_2^0} ds \\
&+7M^2\left\|B\right\|^2 T\int_0^t \mathbf{E}\left\|u_x(s)\right\|^2 ds
\end{aligned}$$

Hence, from (H2) and (H7), combined with property (2) and inequality (18), we have

$$\begin{aligned}
\mathbf{E}\|(\Psi x)(t)\|^2 &\leq 7M^2\mathbf{E}\|x_0\|^2 + 14M^2(\mathbf{E}\|x_{00}\|^2 + \mathbf{E}\|h(0,x_0)\|^2) \\
&\quad + 7M^2T^2(C_h + C_f)\left(1 + \|x\|^2_{\Delta_2^T}\right) + 7M^2TC_g\left(1 + \|x\|^2_{\Delta_2^T}\right) \\
&\quad + 14M^2HT^{2H-1}TL + 7M^2\|B\|^2T^2M_u\left(1 + \|x\|^2_{\Delta_2^T}\right) \\
&\leq 7M^2\left(\mathbf{E}\|x_0\|^2 + 2(\mathbf{E}\|x_{00}\|^2 + \mathbf{E}\|h(0,x_0)\|^2) + 2HT^{2H-1}TL\right) \\
&\quad + 7M^2\left(\|B\|^2T^2(C_h + C_f + M_u) + TC_g\right)\left(1 + \|x\|^2_{\Delta_2^T}\right) \\
&= c_1 + c_2\|x\|^2_{\Delta_2^T}
\end{aligned}$$

where $c_1 \geq 0$ and $c_2 \geq 0$ are suitable constants. Therefore, we obtain that $\|(\Psi x)\|^2_{\Delta_2^T} < \infty$. Since $(\Psi x)(t)$ is continuous on $[0,T]$ and so Ψ maps Δ_2^T into itself.

Step 3 : Ψ is a contraction mapping in Δ_2^T. Let $x, y \in \Delta_2^T$, then for any fixed $t \in [0,T]$ we have

$$\begin{aligned}
\mathbf{E}\|(\Psi x)(t) - (\Psi y)(t)\|^2 &\leq 4\mathbf{E}\left\|\int_0^t S(t-s)B(u_x(s) - u_y(s))\,ds\right\|^2 \\
&\quad + 4\mathbf{E}\left\|\int_0^t C(t-s)(h(s,x(s)) - h(s,y(s)))\,ds\right\|^2 \\
&\quad + 4\mathbf{E}\left\|\int_0^t S(t-s)(f(s,x(s)) - f(s,y(s)))\,ds\right\|^2 \\
&\quad + 4\mathbf{E}\left\|\int_0^t S(t-s)(g(s,x(s)) - g(s,y(s)))\,dw(s)\right\|^2
\end{aligned}$$

By property (2), combined with Hölder's inequality and Ito isometry theorem, we get that

$$\begin{aligned}
\mathbf{E}\|(\Psi x)(t) - (\Psi y)(t)\|^2 &\leq 4M^2\|B\|^2T\int_0^t \mathbf{E}\|u_x(s) - u_y(s)\|^2\,ds \\
&\quad + 4M^2T\int_0^t \mathbf{E}\|h(s,x(s)) - h(s,y(s))\|^2\,ds \\
&\quad + 4M^2T\int_0^t \mathbf{E}\|f(s,x(s)) - f(s,y(s))\|^2\,ds \\
&\quad + 4M^2\int_0^t \mathbf{E}\|g(s,x(s)) - g(s,y(s))\|^2_{\mathcal{L}_2}\,ds
\end{aligned}$$

From (H7), (H8) and property (2), combined with Hölder's inequality and Ito isometry theorem, we have

$$\begin{aligned}
\mathbf{E}\left\|u_x(s)-u_y(s)\right\|^2 &\leq 3\mathbf{E}\left\|\Gamma^{-1}\int_0^T C(T-s)\left[h(s,x(s))-h(s,y(s))\right]ds\right\|^2 \\
&\quad +3\mathbf{E}\left\|\Gamma^{-1}\int_0^t S(T-s)\left[f(s,x(s))-f(s,y(s))\right]ds\right\|^2 \\
&\quad +3\mathbf{E}\left\|\Gamma^{-1}\int_0^t S(T-s)\left[g(s,x(s))-g(s,y(s))\right]dw(s)\right\|^2 \\
&\leq 3M_\Gamma M^2 T(C_h+C_f)\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds \\
&\quad +3M_\Gamma M^2 C_g\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds \\
&\leq 3M_\Gamma M^2\left(T(C_h+C_f)+C_g\right)\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds \\
&= M_\mu\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds
\end{aligned}$$

where $M_\mu = 3M_\Gamma M^2\left(T(C_h+C_f)+C_g\right).$
Therefore,

$$\begin{aligned}
\mathbf{E}\left\|(\Psi x)(t)-(\Psi y)(t)\right\|^2 &\leq 4M^2 T\left(C_h+C_f\right)\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds \\
&\quad +4M^2 C_g\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds \\
&\quad +4M^2\left\|B\right\|^2 T^2 M_\mu\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds
\end{aligned}$$

Hence, we obtain a positive real constant $\gamma(T)$ such that

$$\mathbf{E}\left\|(\Psi x)(t)-(\Psi y)(t)\right\|^2 \leq \gamma(T)\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds$$

where

$$\gamma(T) = 4M^2 T\left(\left\|B\right\|^2 T M_\mu + T\left(C_h+C_f\right)+C_g\right).$$

Moreover

$$\begin{aligned}
\mathbf{E}\left\|(\Psi^2 x)(t)-(\Psi^2 y)(t)\right\|^2 &\leq \gamma(T)\int_0^t \mathbf{E}\left\|(\Psi x)(s)-(\Psi y)(s)\right\|^2 ds \\
&\leq \gamma(T)t\int_0^t \gamma(T)\mathbf{E}\left\|x(s)-y(s)\right\|^2 ds \\
&= (\gamma(T))^2\, t\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds
\end{aligned}$$

For any natural number n, using mathematical induction, one can get

$$\begin{aligned}\mathbf{E}\left\|(\Psi^n x)(t)-(\Psi^n y)(t)\right\|^2 &\leq \gamma(T)\int_0^t \mathbf{E}\left\|(\Psi^{n-1}x)(s)-(\Psi^{n-1}y)(s)\right\|^2 ds\\ &\leq \frac{(t\gamma(T))^n}{n!}\left\|x-y\right\|^2_{\Delta_2^t}\end{aligned}$$

Then taking the supremum over $[0,T]$,

$$\left\|(\Psi^n x)(t)-(\Psi^n y)(t)\right\|^2_{\Delta_2^T} \leq \frac{(T\gamma(T))^n}{n!}\left\|x-y\right\|^2_{\Delta_2^T}$$

For sufficiently large n we have $\frac{(T\gamma(T))^n}{n!}<1$. It follows that Ψ^n is a strict contraction mapping on Δ_2^T, so that The Banach fixed point theorem ensure that Ψ has a unique fixed point, which is a mild solution for (15). Which implies that the system (15) is controllable on $[0,T]$.

Remark 3 *The theory of impulsive differential equations has found enormous applications in the realistic mathematical modeling of a wide range of practical situations, many systems in physics and biology exhibit impulsive dynamical behavior because of sudden jumps at certain instants in the evolution process. A lot of dynamic systems have variable structures subjects to stochastic abrupt changes resulting from abrupt phenomena. For some recent works on the existence and controllability results of impulsive stochastic differential equations, we refer the reader to monographs ([24], [25], [26], [27], [28]).In this remark we consider the following system*

$$\begin{cases} d\left(x^{'}(t)-h(t,x(t))\right)=Ax(t)dt+Bu(t)dt+f\left(t,x(t)\right)dt+g(t,x(t)dw(t)+\sigma(t)dZ_H(t),\\ x(0)=x_0, \quad x^{'}(0)=x_{00}, \quad t\in[0,T], \quad t\neq t_k\\ \Delta x(t_k)=x(t_k^+)-x(t_k^-)=I_k(x(t_k^-)), \quad k=1,2,...,m, \end{cases} \tag{19}$$

where $h,\ f,\ g,\ \sigma,\ A,\ B$ are the same as in the Eq.(15), The fixed moments of times t_k satisfies $0=t_0<t_1<...<t_m<t_{m+1}=T$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t=t_k$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$ represents the jump in the state x at time t_k, where $I_k:X\to X$ determines the size of the jump. The mild solution of (19) *is given by*

$$\begin{aligned}x(t) &= C(t)x_0+S(t)(x_{00}-h(0,x_0))+\int_0^t C(t-s)h(s,x(s))ds\\ &+\int_0^t S(t-s)\left(Bu(s)+f(s,x(s)\right)ds+\int_0^t S(t-s)g(s,x(s))dw(s)\\ &+\int_0^t S(t-s)\sigma(s)dZ_H(s), \quad P-a.s.\end{aligned}$$

If I_k satisfies Lipschitz and linear growth conditions, by adopting the method used in Theorem (2) we prove the controllability of system (19) on $[0,T]$.

5 Example

Consider the control system driven by the process w and Z_H to illustrate the obtained theory

$$\begin{cases} \partial\left[\frac{\partial x(t,z)}{\partial t} - h_1\left(t, x(t,z)\right)\right] = \frac{\partial^2}{\partial z^2}x(t,z)\partial t + \left(v(t,z) + f_1\left(t, x(t,z)\right)\right)\partial t \\ +g_1\left(t, x(t,z)\right)dw(t) + \sigma(t)dZ_H,\ t \in]0,T[,\ z \in [0,\pi] \\ x(0,z) = x_0(z), \quad z \in [0,\pi], \\ \frac{\partial x(0,z)}{\partial t} = x_{00}(z), \quad z \in [0,\pi], \\ x(t,0) = x(t,\pi) = 0, \quad t \in [0,T]\,. \end{cases} \tag{20}$$

Let $X = K = Y = U = L_2[0,\pi]$ and $x_0,\ x_{00} \in L_2[0,\pi]$. Let $A \subset D(A) : X \to X$ be the linear operator given by $Ay = y^{''}$, where

$$D(A) = \{y \in X\ /\ y,\ y^{'} \text{ are absolutely continuous } y^{''} \in X, \quad y(0) = y(\pi) = 0\}.$$

$w(t)$ denotes a one dimensional standard Brownian motion and Z_H is a Rosenblatt, the process w and Z_H are independent. Suppose $h_1, f_1, g_1 : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are continuous, satisfy Lipschitz condition and linear growth condition and uniformly bounded.

First of all, note that there exists a complete orthonormal set $\{e_n\}_{n\geq 1}$ of eigenvectors of A with

$$e_n(z) = \sqrt{(2/\pi)}\sin nz,\ 0 \leq z \leq \pi,\ n = 1, 2, ...$$

and the following properties hold

i) If $y \in D(A)$, then

$$Ay = -\sum_{n=1}^{\infty} n^2 \langle y, e_n\rangle\, e_n(y), \quad y \in D(A),$$

ii) The operator $C(t)$ defined by

$$C(t)y = \sum_{n=1}^{\infty} \cos(nt)\langle y, e_n\rangle\, e_n, \quad y \in X$$

is the cosine family in X generated by $(A, D(A))$, and the associated sine family is given by

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}\langle y, e_n\rangle\, e_n, \quad y \in X.$$

It is clear that $C(.)x$ and $S(.)x$ are periodic functions, and $\|C(t)\| \leq 1,\ \|S(t)\| \leq 1,\ t \in \mathbb{R}$.

Now define the functions: $h,\ f : [0,T] \times X \to X$, and $g : [0,T] \times X \to \mathcal{L}_2(K,X)$ as follows

$$\begin{array}{l} h(t,x)(z) = h_1(t, x(z)), \\ f(t,x)(z) = f_1(t, x(z)), \\ g(t,x)(z) = g_1(t, x(z)) \end{array}$$

for $t \in [0,T],\ x \in X$ and $0 < z < \pi$. The function $\sigma : [0,T] \to \mathcal{L}_2^0(Y,X)$ is bounded.

Let $B : U \to X$ is a bounded linear operator defined by

$$Bu(t)(z) = v(t, z),\ 0 \leq z \leq \pi,\ u \in L^2([0, T], U)$$

The operator $\Gamma : L^2([0, T], U) \to X$ given by

$$\Gamma u = \int_0^T S(T - s)v(s, z)ds.$$

Γ is a bounded linear operator but not necessarily one-to-one.Let

$$\ker(\Gamma) = \{x \in L^2([0, T], U),\ \Gamma x = 0\}$$

be the null space of Γ and $[\ker(\Gamma)]^{\perp}$ be its orthogonal complement in $L^2([0, T], U)$. Let $\widetilde{\Gamma}$: $[\ker(\Gamma)]^{\perp} \to Range(\Gamma)$ be the restriction of Γ to $[\ker(\Gamma)]^{\perp}$, $\widetilde{\Gamma}$ is necessarily one-to-one operator. The inverse mapping theorem says that $\widetilde{\Gamma}^{-1}$ is bounded since $[\ker(\Gamma)]^{\perp}$ and $Range(\Gamma)$ are Banach spaces. So That Γ^{-1} is bounded and takes values in $L^2([0, T], U) \perp \ker(\Gamma)$, hypothesis (H4) is satisfied. Hence, all conditions of Theorem(2) are satisfied, and consequently system.(20) is controllable on $[0, T]$.

References

[1] Nazim I Mahmudov. Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces. *SIAM journal on control and optimization*, 42(5):1604–1622, 2003.

[2] Jerzy Klamka. *Controllability and Minimum Energy Control.* Springer, 2019.

[3] Malik Muslim, Avadhesh Kumar, and Michal Fečkan. Existence, uniqueness and stability of solutions to second order nonlinear differential equations with non-instantaneous impulses. *Journal of King Saud University-Science*, 30(2):204–213, 2018.

[4] Xiaofeng Su and Xianlong Fu. Approximate controllability of second-order stochastic differential systems driven by a lévy process. *Applied Mathematics & Optimization*, pages 1–27, 2019.

[5] Jerzy Klamka. Constrained controllability of second order dynamical systems with delay. In *Controllability and Minimum Energy Control*, pages 67–88. Springer, 2019.

[6] Jan Gairing, Peter Imkeller, Radomyra Shevchenko, and Ciprian Tudor. Hurst index estimation in stochastic differential equations driven by fractional brownian motion. *Journal of Theoretical Probability*, pages 1–24, 2019.

[7] Hamdy M Ahmed, Mahmoud M El-Borai, and M Elsaid Ramadan. Boundary controllability of nonlocal hilfer fractional stochastic differential systems with fractional brownian motion and poisson jumps. *Advances in Difference Equations*, 2019(1):1–23, 2019.

[8] Zhi Li, Yuan Yuan Jing, and Liping Xu. Controllability of neutral stochastic evolution equations driven by fbm with hurst parameter less than 1/2. *International Journal of Systems Science*, 50(9):1835–1846, 2019.

[9] Murray Rosenblatt. Independence and dependence. In *Proc. 4th Berkeley sympos. math. statist. and prob*, volume 2, pages 431–443, 1961.

[10] Murad Taqqu. Weak convergence to fractional brownian motion and to the rosenblatt process. *Advances in Applied Probability*, 7(2):249–249, 1975.

[11] NN Leonenko and VV Anh. Rate of convergence to the rosenblatt distribution for additive functionals of stochastic processes with long-range dependence. *Journal of applied mathematics and stochastic analysis*, 14, 1970.

[12] Guangjun Shen, R Sakthivel, Yong Ren, and Mengyu Li. Controllability and stability of fractional stochastic functional systems driven by rosenblatt process. *Collectanea Mathematica*, 71(1):63–82, 2020.

[13] S Saravanakumar and P Balasubramaniam. On impulsive hilfer fractional stochastic differential system driven by rosenblatt process. *Stochastic Analysis and Applications*, 37(6):955–976, 2019.

[14] Ankit Kumar, Ramesh K Vats, and Avadhesh Kumar. Approximate controllability of second-order non-autonomous system with finite delay. *Journal of Dynamical and Control Systems*, pages 1–17, 2020.

[15] T Sathiyaraj, J Wang, and D O Regan. Controllability of stochastic nonlinear oscillating delay systems driven by the rosenblatt distribution. In *Proc. R. Soc. Edinburgh Sect. A*, 2020.

[16] El Hassan Lakhel and MA McKibben. Controllability for time-dependent neutral stochastic functional differential equations with rosenblatt process and impulses. *International Journal of Control, Automation and Systems*, 17(2):286–297, 2019.

[17] R Sakthivel, P Revathi, Yong Ren, and Guangjun Shen. Retarded stochastic differential equations with infinite delay driven by rosenblatt process. *Stochastic analysis and applications*, 36(2):304–323, 2018.

[18] Bin Pei and Yong Xu. Mild solutions of local non-lipschitz neutral stochastic functional evolution equations driven by jumps modulated by markovian switching. *Stochastic Analysis and Applications*, 35(3):391–408, 2017.

[19] Abbes Benchaabane and Rathinasamy Sakthivel. Sobolev-type fractional stochastic differential equations with non-lipschitz coefficients. *Journal of Computational and Applied Mathematics*, 312:65–73, 2017.

[20] Hector O Fattorini. *Second order linear differential equations in Banach spaces*. Elsevier, 1985.

[21] Ciprian A Tudor. Analysis of the rosenblatt process. *ESAIM: Probability and statistics*, 12:230–257, 2008.

[22] Guangjun Shen and Yong Ren. Neutral stochastic partial differential equations with delay driven by rosenblatt process in a hilbert space. *Journal of the Korean Statistical Society*, 44(1):123–133, 2015.

[23] Dorel Barbu. Local and global existence for mild solutions of stochastic differential equations. *Portugaliae Mathematica*, 55(4):411–424, 1998.

[24] A Anguraj, K Ravikumar, and Dumitru Baleanu. Approximate controllability of a semilinear impulsive stochastic system with nonlocal conditions and poisson jumps. *Advances in Difference Equations*, 2020(1):65, 2020.

[25] Liguang Xu, Shuzhi Sam Ge, and Hongxiao Hu. Boundedness and stability analysis for impulsive stochastic differential equations driven by g-brownian motion. *International Journal of Control*, 92(3):642–652, 2019.

[26] Hao Huang, Zheng Wu, Ling Hu, Zhangzhi Wei, and Lianglong Wang. Existence and controllability of second-order neutral impulsive stochastic evolution integro-differential equations with state-dependent delay. *Journal of Fixed Point Theory and Applications*, 20(1):9, 2018.

[27] Qian Chen, Amar Debbouche, Zijian Luo, and JinRong Wang. Impulsive fractional differential equations with riemann–liouville derivative and iterative learning control. *Chaos, Solitons & Fractals*, 102:111–118, 2017.

[28] Abbes Benchaabane. Complete controllability of general stochastic integrodifferential systems. *MATHEMATICAL REPORTS*, 18(4):437–448, 2016.