Global attractors of the periodic initial value problem for Landau–Lifshitz–Bloch–Maxwell system

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Abstract. This paper is devoted to study the global attractors of the periodic initial value problem for Landau–Lifshitz–Bloch–Maxwell system. Fist we give the global existence of the smooth solution for this system. Then, we prove the existence of global attractors, the Hausdorff dimension and fractal dimension have been estimated.

Keywords: Landau–Lifshitz–Bloch–Maxwell system, Smooth solutions, Global attractors.

1 Introduction and main results

The purpose of the present paper is to consider the following periodic initial value problem for the Landau–Lifshitz–Bloch–Maxwell system

$$Z_t = \Delta Z + Z \times (\Delta Z + H) - k(1 + \mu|Z|^2)Z, \tag{1.1}$$

$$\nabla \times H = E_t + \sigma E, \tag{1.2}$$

$$\nabla \times E = -H_t - \beta Z_t - \delta H, \tag{1.3}$$

$$\nabla \cdot H + \beta \nabla \cdot Z = 0, \ \nabla \cdot E = 0, \tag{1.4}$$

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with periodic conditions

$$Z(x+2Le_i,t)=Z(x,t),\ H(x+2Le_i,t)=H(x,t),$$
$$E(x+2Le_i,t)=E(x,t),\ x\in\Omega,\ t\geq 0,\ i=1,2,3, \tag{1.5}$$
$$Z(x,0)=Z_0(x),\ H(x,0)=H_0(x),\ E(x,0)=E_0(x),\ x\in\Omega, \tag{1.6}$$

where $Z=(Z_1,Z_2,Z_3)$ is the vector of magnetization, $H=(H_1,H_2,H_3)$ is the magnetic field, $E=(E_1,E_2,E_3)$ is the electric field. $x+2Le_i=(x_1,\cdots,x_{i-1},x_i+2L,x_{i+1},\cdots,x_n)$, $i=1,2,3,\ L>0,\ k>0,\ \mu>0,\ \sigma>0,\ \delta>0,\ \beta>0.$

System (1.1)-(1.4) arises in the study of the Landau–Lifshitz–Bloch (LLB) system which well describes the magnetization dynamics of ferromagnets at high temperature ($\theta\geq\theta_c$, θ_c–Curie value). Before that, the first thing people studied was the low temperature ($\theta\leq\theta_c$) which was called the Landau–Lifshitz (LL) system. The evolution of spin fields in continuum ferromagnets around the effective field H_{eff} is described by the Landau–Lifshitz-Gilbert (LLG) equation as follows:

$$Z_t=\alpha_1 Z\times H_{eff}-\alpha_2 Z\times(Z\times H_{eff}), \tag{1.7}$$

where $Z(x,t)=(Z_1(x,t),Z_2(x,t),Z_3(x,t))$ is magnetization functional vector. $\alpha_1,\alpha_2>0$. "$\times$" denotes the vector outer product. The saturation magnetization $|Z|\equiv 1$ and the effective field $H_{eff}=-\frac{\delta\varepsilon_\alpha(Z)}{\delta Z}$. The Landau energy

$$\varepsilon_\alpha(Z)=\alpha\int_\Omega|\nabla Z|^2dx+\int_\Omega\varphi(Z)dx+\frac{1}{2}\int_{R^n}|\nabla u|^2dx-\int_\Omega<H,Z>_{R^n}dx,$$

where $\alpha\int_\Omega|\nabla Z|^2dx$ is the exchange energy with $\alpha>0$, $\int_\Omega\varphi(Z)dx$ is the anisotropy energy, $\frac{1}{2}\int_{R^n}|\nabla u|^2dx$ is the magnetostatic energy from the stray field, $-\nabla u$ satisfies $div(-\mu_0\nabla u+Z\chi_\Omega)=0$ in R^n for the vacuum permeability $\mu_0=1$ and χ_Ω is the characteristic function. It is widely known that the LLG equation is hard to analyze mathematically because of its strongly coupled degenerated quasi-linear parabolic system. So many researchers usually restrict the exchange of energy contributions in the effective field as the reduced effective field $H_{eff}=\Delta Z$, and many important results have been obtain. A.Visintin gave the first existence results for the LLG equation in [2]. In [3] and [4], the authors prove the globally existence solutions via the valid energy law. For more relevant results, one can see [1] and the literatures in it.

However, there is an interesting phenomenon. The LLG equation only gives a good description about the magnetization dynamics of ferromagnets at low temperatures, when the temperature is higher than the Curie temperature (Curie value θ_c), the modulus of magnetization change and the LLG equation is dissatisfactory. From the physical background, materials are classified as paramagnetic or ferromagnetic according to there different reactions to the presence of an external magnetic field. This reaction can be affected by many different factors such as temperature. When the temperature overcome as the critical value θ_c, the material is paramagnetic [5].

In order to describe the dynamics of magnetization vector Z in a ferromagnetic body for a wide range of temperatures, in 1990 Garanin [8,9] derived the Landau–Lifshitz–Bloch

(LLB) equation from statistical mechanis with the mean field approximation. However, at high temperature ($\theta \geq \theta_c$, θ_c–Curie value), LLB model is satisfactory. In [10], A. Berti et. al. also pointed that from the paramagnetic to the ferromagnetic state is modeled as a second order phase transition. It is necessary to consider equation of the temperature θ.

The LLB equation is given as follows

$$Z_t = -\gamma Z \times H_{eff} + \frac{L_1}{|Z|^2}(Z \cdot H_{eff})Z - \frac{L_2}{|Z|^2} Z \times (Z \times H_{eff}), \tag{1.8}$$

where γ are constants, L_1, L_2 are the longitudinal and transverse damping parameters depend on the temperature. H_{eff} is the effective field. By using method in [6] [7], We can also rewrite (1.8) as follows

$$Z_t = -\gamma Z \times H_{eff} + \frac{\gamma a_{\|}}{|Z|^2} - \frac{\gamma a_{\perp}}{|Z|^2} Z \times (Z \times H_{eff}) \tag{1.9}$$

where $\gamma a_{\|} = L_1$, $\gamma a_{\perp} = L_2$. Here $a_{\|}$ and $a_{\perp}$ are dimensionless damping parameters depend on the temperature. It can be write as follows [11]

$$a_{\|}(\theta) = \frac{2\theta}{3\theta_c}\lambda, \quad a_{\perp}(\theta) = \begin{cases} \lambda\left(1 - \dfrac{\theta}{3\theta_c}\right), & \text{if } \theta < \theta_c \\ a_{\|}(\theta), & \text{if } \theta \geq \theta_c, \end{cases}$$

where $\lambda > 0$ is a constant. From the calculations above, we can see that when the temperature is greater than θ_c, we obtain $\alpha_{\|} = \alpha_{\perp}$. We can rewrite (1.8) as

$$Z_t = -\gamma Z \times H_{eff} + \gamma \alpha_{\perp} H_{eff}$$

by using the vector triple product.

In [12], the author get

$$Z \times (Z \times H_{eff}) = (Z \cdot H_{eff})Z - |Z|^2 H_{eff}$$

according to the vector triple product identity

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$$

moreover points that if $L_1 = L_2$, (1.8) can be reduced as follows

$$Z_t = \Delta Z + Z \times \Delta Z - k|Z|^2 Z, \quad (k > 0) \tag{1.10}$$

and the existence of weak solution for the equation (1.10) has been obtained.

For the convenience of formulating our results, we set some notations on the functional settings. We give function spaces and some notation. $L^2(\Omega)$ represents the Hilbert space equipped with the inner product

$$(u, v) = \int_\Omega u\bar{v}dx, \quad ||u|| = \sqrt{(u, u)},$$

to describe the theorems accurately , we denote the $L^2(\Omega)$-norm by $||\cdot||_2$, denote the $L^p(\Omega)$-norm by $||\cdot||_p$. $H^\sigma(\Omega)$ represents the sobolev space $\{u\in L^2(\Omega), D^K u\in L^2(\Omega), K\leq\sigma\}$, and as usual, $H^K(\Omega)=W^{K,2}(\Omega)$. $C^K(I,\Omega)$ represents the space of continuous functions from the interval I to a Banach space Ω. Throughout this paper, the Sobolev embedding theorem will be used frequently. We choose the definition domains of the Laplacian $\triangle$ as follows:

$$D(\triangle)=H^2(\Omega)$$

We can essentially prove the following result with generalizes the work given by Galerkin's approximation in [1].

Theorem 1.1 *(Local existence of the smooth solution)*
Assume the initial data $(Z_0(x),H_0(x),E_0(x))\in(H^k(\Omega),H^{k-1}(\Omega),H^{k-1}(\Omega))$, $k\geq 1+[\frac{d}{2}]$, $\Omega\subset\mathbb{R}^d$, $d=1,2$. $\nabla\cdot E_0=0$, $\nabla\cdot(H_0+\beta Z_0)=0$. *Then there exists a constant* $T_0>0$ *such that the periodic initial value problem* (1.1)-(1.6) *admits a local smooth solution*

$$Z(x,t)\in\cap_{s=0}^{[\frac{k}{2}]}W_\infty^s(0,T_0;H^{k-2s}(\Omega)), \tag{1.11}$$
$$H(x,t)\in\cap_{s=0}^{k-1}W_\infty^s(0,T_0;H^{k-s-1}(\Omega)), \tag{1.12}$$
$$E(x,t)\in\cap_{s=0}^{k-1}W_\infty^s(0,T_0;H^{k-s-1}(\Omega)). \tag{1.13}$$

It is natural to consider the global existence of the smooth solution, and the first result of the present paper is stated in the following theorem.

Theorem 1.2 *(Global existence of the smooth solution)*
Assume that the conditions in Lemma 2.4 are satisfied, and the initial data $(Z_0(x),H_0(x),E_0(x))\in(H^k(\Omega),H^{k-1}(\Omega),H^{k-1}(\Omega))$, $k\geq 1+[\frac{d}{2}]$, $\Omega\subset\mathbb{R}^d$ *is a bounded domain and* $1\leq d\leq 2$. $\nabla\cdot E_0=0$, $\nabla\cdot(H_0+\beta Z_0)=0$. *When* $d=2$, *assume that* $\|Z_0(x)\|_{H^2(\Omega)}<\delta_1$, *where* δ_1 *is a small constant, then there exists a unique global smooth solution of the periodic initial value problem* (1.1)-(1.6) *satisfies*

$$Z(x,t)\in\cap_{s=0}^{[\frac{k}{2}]}W_\infty^s(0,T;H^{k-2s}(\Omega)), \tag{1.14}$$
$$H(x,t)\in\cap_{s=0}^{k-1}W_\infty^s(0,T;H^{k-s-1}(\Omega)), \tag{1.15}$$
$$E(x,t)\in\cap_{s=0}^{k-1}W_\infty^s(0,T;H^{k-s-1}(\Omega)). \tag{1.16}$$

In the following, we denote D by the set of smooth solutions of (1.1)-(1.6), d represents the metric of D. The semigroup operator which is continuous of (1.1)-(1.6) is given by

$$S(\tau):D\to D, S(\tau+T)=S(\tau)\cdot S(T),(\forall\tau,T\geq 0), S(0)=I.$$

Definition 1.1 $B_0\subset D$ *is an attracting set, if for all bounded sets* $B\subset D$ *we have* $d(S(\tau)B,B_0)\to 0(\tau\to\infty)$, *where* $d(A,B)$ *is the semidistance defined by*

$$d(A,B)=sup_{x\in A}inf_{y\in B}d(x,y)$$

.

Definition 1.2 *Define the $\omega-$limit set of a bounded attracting set $B_0 \subset D$ at time τ as*

$$A_0 = \omega(B_0) = \bigcap_{s\geq 0} \overline{\bigcup_{\tau \geq s} S(\tau) B_0}$$

.

Definition 1.3 *The set $A \subset D$ is called a global attractor if*
(i). A is compact and invariant: $S(t)A = A$ for all $t \geq 0$;
(ii). $dist(S(t)B, A) \to 0 (t \to \infty)$ for every bounded set $B \subset D$.

Then the second result of the present paper is stated in the following theorem.

Theorem 1.3 *Assume that the conditions of Theorem 1.2 hold. Then there exists an attractor $\mathcal{A}$ of the periodic initial value problem* (1.1)-(1.6) *satisfies*
(i) $\mathcal{A}$ is weakly compact in $H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega)$,
(ii) $S(t)\mathcal{A} = \mathcal{A}$,
(iii) $\lim_{t\to\infty} dist(S(t)B, \mathcal{A}) = 0$ for any bounded set $B \subset D \subset H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega)$, where

$$D = \{(Z, E, H) \in H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega), \nabla \cdot E = 0, \nabla \cdot (H + \beta Z) = 0\},$$

$$dist(x, y) = \sup_{x\in X} \inf_{y \in Y} \|x - y\|.$$

$S(t)(Z_0, H_0, E_0)$ is the semigroup formed by problem (1.1)-(1.6).

Moreover, we estimate the Hausdorff dimension and fractal dimension in Theorem 3.1 in the end of paper.
The rest of this paper is organized as follows. In section 2, we give the proof of the existence of the global smooth solution. In section 3, we prove the second main result, Theorem 1.3. And the Hausdorff dimension and fractal dimension are also estimated in this section.

2 Global existence of the smooth solution

Lemma 2.1 *Assume that $Z_0(x) \in L^2(\Omega)$. Then for the smooth solution of the periodic initial value problem* (1.1)-(1.6)*, there are*

$$\|Z(\cdot, t)\|_2^2 \leq \|Z_0(x)\|_2^2. \tag{2.1}$$

Proof. Making the scalar product of Z with (1.1), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Z(\cdot, t)\|_2^2 + \|\nabla Z\|_2^2 + k\int_\Omega (1 + \mu|Z|^2)|Z|^2 \mathrm{d}x = 0.$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Z(\cdot, t)\|_2^2 \leq 0.$$

As a result, (2.1) follows.

Lemma 2.2 *Assume $Z_0(x) \in H^2(\Omega)$, then we have*

$$\sup_{0\le t<\infty} \|Z(\cdot,t)\|_{L^\infty} \le \|Z_0(x)\|_{H^2}. \tag{2.2}$$

Proof. A direct calculation yields

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|Z(\cdot,t)\|_p^p &= \int_\Omega |Z|^{p-2}Z\cdot Z_t\mathrm{d}x = \int_\Omega |Z|^{p-2}[Z\cdot\Delta Z - k(1+\mu|Z|^2)|Z|^2]\mathrm{d}x \\ &= -\int_\Omega \nabla(|Z|^{p-2}Z)\cdot\nabla Z - k\int_\Omega (1+\mu|Z|^2)|Z|^p\mathrm{d}x \le 0.\end{aligned}$$

Thus, we get

$$\|Z(\cdot,t)\|_p^p \le \|Z_0(x)\|_p^p.$$

Letting $p\to\infty$, we get (2.2).

Lemma 2.3 *Assume $\nabla Z_0(x) \in L^2(\Omega)$, $E_0(x) \in L^2(\Omega)$, $H_0(x) \in L^2(\Omega)$. Then for the smooth solution of the periodic initial value problem* (1.1)-(1.6)*, we have*

$$\sup_{0\le t<\infty}[\|\nabla Z(\cdot,t)\|_2^2 + \|E(\cdot,t)\|_2^2 + \|H(\cdot,t)\|_2^2] \le K_1, \tag{2.3}$$

$$\int_0^\infty \|\Delta Z(\cdot,t)\|_2^2 dt \le K_2, \tag{2.4}$$

where the constants K_1 and K_2 depend only $\|\nabla Z_0(x)\|_2$, $\|E_0(x)\|_2$, $\|H_0(x)\|_2$.

Proof. Making the scalar product of E with (1.2), and making the scalar product of $-H$ with (1.3), and then adding these two equalities obtained, we have

$$(\nabla\times H)\cdot E - (\nabla\times E)\cdot H = E_t\cdot E + \sigma|E|^2 + H_t\cdot H + \beta Z_t\cdot H + \delta|H|^2. \tag{2.5}$$

By using the formula

$$(\nabla\times H)\cdot E - (\nabla\times E)\cdot H = \nabla\cdot(H\times E) \tag{2.6}$$

and integrating (2.5) with respect to x over Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|E\|_2^2 + \|H\|_2^2) + \sigma\|E\|_2^2 + \delta\|H\|_2^2 + \beta\int_\Omega Z_t\cdot H\mathrm{d}x = 0. \tag{2.7}$$

Making the scalar product of $(\Delta Z + H)$ with (1.1) and integrating with respect to x over Ω, we get

$$\begin{aligned}\int_\Omega Z_t\cdot H\mathrm{d}x = &-\int_\Omega \Delta Z\cdot Z_t\mathrm{d}x + \int_\Omega |\Delta Z|^2\mathrm{d}x + \int_\Omega \Delta Z\cdot H\mathrm{d}x \\ &-k\int_\Omega (1+\mu|Z|^2)Z\cdot\Delta Z\mathrm{d}x - k\int_\Omega (1+\mu|Z|^2)Z\cdot H\mathrm{d}x.\end{aligned} \tag{2.8}$$

Applying Young's inequality, we have

$$\left|\int_\Omega \Delta Z\cdot H\mathrm{d}x\right|\le \frac{1}{2}(\|\Delta Z\|_2^2+\|H\|_2^2),$$

$$\left|k\int_\Omega (1+\mu|Z|^2)Z\cdot \Delta Z\mathrm{d}x\right|\le \frac{1}{3}\|\Delta Z\|_2^2+c_1\|Z_0(x)\|_{H^2}^2,$$

$$\left|k\int_\Omega (1+\mu|Z|^2)Z\cdot H\mathrm{d}x\right|\le \frac{1}{3}\|H\|_2^2+c_2\|Z_0(x)\|_{H^2}^2.$$

From (2.7), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|E\|_2^2+\|H\|_2^2+\beta\|\nabla Z\|_2^2)+\sigma\|E\|_2^2+\frac{\beta}{6}\|\Delta Z\|_2^2+(\delta-\frac{5\beta}{6})\|H\|_2^2\le c_3. \tag{2.9}$$

By using Gronwall inequality, we can get (2.3) and (2.4).

In order to get the uniform estimates with respect to t for the solution $(Z,E,H)\in(H^2(\Omega),H^1(\Omega),H^1(\Omega))$, we first rewrite (1.2) and (1.3) as the following equivalent second order nonlinear wave equations

$$\nabla\times(\nabla\times H)=\frac{\partial}{\partial t}(\nabla\times E)+\sigma\nabla\times E, \tag{2.10}$$

$$\nabla\times(\nabla\times E)=-\frac{\partial}{\partial t}(\nabla\times H)-\beta\frac{\partial}{\partial t}(\nabla\times Z)-\delta\nabla\times H. \tag{2.11}$$

From the formula

$$\nabla\times(\nabla\times H)=\nabla(\nabla\cdot H)-\Delta H=-\beta\nabla(\nabla\cdot Z)-\Delta H, \tag{2.12}$$

we have

$$\begin{aligned}-\beta\nabla(\nabla\cdot Z)-\Delta H&=\frac{\partial}{\partial t}(\nabla\times E)+\sigma\nabla\times E\\&=-H_{tt}-\beta Z_{tt}-(\delta+\sigma)H_t-\beta\sigma Z_t-\sigma\delta H,\end{aligned}$$

$$\begin{aligned}-\Delta E&=-\frac{\partial}{\partial t}(\nabla\times H)-\beta\frac{\partial}{\partial t}(\nabla\times Z)-\delta\nabla\times H\\&=-E_{tt}-(\sigma+\delta)E_t-\delta\sigma E-\beta(\nabla\times Z)_t.\end{aligned}$$

Thus, we find

$$H_{tt}-\Delta H+\beta Z_{tt}+(\delta+\sigma)H_t+\beta\sigma Z_t+\sigma\delta H-\beta\nabla(\nabla\cdot Z)=0, \tag{2.13}$$

$$E_{tt}-\Delta E+(\sigma+\delta)E_t+\delta\sigma E+\beta(\nabla\times Z)_t=0. \tag{2.14}$$

It is difficult to derive the uniform a priori estimates with respect to t for $(Z,E,H)\in(H^2(\Omega),H^1(\Omega),H^1(\Omega))$ from (2.13) and (2.14). We prove instead by Lyapunov functional containing small parameter method. We define the Lyapunov functional as follows:

$$e(t)=\frac{1}{2}(\|H_t(\cdot,t)\|_2^2+\|\nabla H(\cdot,t)\|_2^2+\|E_t(\cdot,t)\|_2^2+\|\nabla E\|_2^2+\|\Delta Z\|_2^2)+\eta_1(H,H_t)+\eta_2(E,E_t), \tag{2.15}$$

where η_1 and η_2 are constants to be determined. We want to prove that $e(t)$ satisfies the following differential inequality

$$\frac{\mathrm{d}e(t)}{\mathrm{d}t} + ae(t) \leq K, \tag{2.16}$$

where $a > 0$ is a constant, K independent of t. Then the a priori estimates can be derived.

In fact, it follows from (2.15) that

$$\begin{aligned}\frac{\mathrm{d}e(t)}{\mathrm{d}t} &= (H_t, H_{tt}) + (\nabla H, \nabla H_t) + (E_t, E_{tt}) + (\nabla E, \nabla E_t) + (\Delta Z, \Delta Z_t) \\ &\quad + \eta_1(H_t, H_t) + \eta_1(H, H_{tt}) + \eta_2(E_t, E_t) + \eta_2(E, E_{tt}),\end{aligned} \tag{2.17}$$

in which

$$\begin{aligned}(H_t, H_{tt}) &= (H_t, \Delta H - \beta Z_{tt} - (\delta+\sigma)H_t - \beta\sigma Z_t - \sigma\delta H + \beta\nabla(\nabla\cdot Z)) \\ &= (H_t, \Delta H) - \beta(H_t, Z_{tt}) - (\delta+\sigma)(H_t, H_t) - \beta\sigma(H_t, Z_t) \\ &\quad - \sigma\delta(H_t, H) + \beta(H_t, \nabla(\nabla\cdot Z)), \\ (\nabla H, \nabla H_t) &= -(\Delta H, H_t), \\ (E_t, E_{tt}) &= (E_t, \Delta E - (\sigma+\delta)E_t - \delta\sigma E - \beta(\nabla\times Z)_t) \\ &= (E_t, \Delta E) - (\sigma+\delta)(E_t, E_t) - \delta\sigma(E_t, E) - \beta(E_t, (\nabla\times Z)_t), \\ (\nabla E, \nabla E_t) &= -(\Delta E, E_t), \\ (\Delta Z, \Delta Z_t) &= (\Delta Z, \Delta^2 Z) + (\Delta Z, \Delta(Z\times\Delta Z)) + (\Delta Z, \Delta(Z\times H)) \\ &\quad - k(\Delta Z, \Delta((1+\mu|Z|^2)Z)),\end{aligned}$$

$$\begin{aligned}\eta_1(H, H_{tt}) &= \eta_1(H, \Delta H - \beta Z_{tt} - (\delta+\sigma)H_t - \beta\sigma Z_t - \sigma\delta H + \beta\nabla(\nabla\cdot Z)) \\ &= -\eta_1\|\nabla H\|_2^2 - \beta\eta_1(H, Z_{tt}) - (\delta+\sigma)\eta_1(H, H_t) - \beta\sigma\eta_1(H, Z_t) \\ &\quad -\sigma\delta\eta_1\|H\|_2^2 + \beta\eta_1(H, \nabla(\nabla\cdot Z)), \\ \eta_2(E, E_{tt}) &= \eta_2(E, \Delta E - (\sigma+\delta)E_t - \delta\sigma E - \beta(\nabla\times Z)_t) \\ &= -\eta_2\|\nabla E\|_2^2 - (\sigma+\delta)\eta_2(E, E_t) - \delta\sigma\eta_2\|E\|_2^2 - \beta\eta_2(E, (\nabla\times Z)_t).\end{aligned}$$

Hence, we have

$$\begin{aligned}\frac{\mathrm{d}e(t)}{\mathrm{d}t} &= -(\delta+\sigma-\eta_1)\|H_t\|_2^2 - (\delta+\sigma-\eta_2)\|E_t\|_2^2 - \eta_1\|\nabla H\|_2^2 - \eta_2\|\nabla E\|_2^2 \\ &\quad -\sigma\delta\eta_1\|H\|_2^2 - \sigma\delta\eta_2\|E\|_2^2 - \beta\eta_1(H, Z_{tt}) - ((\delta+\sigma)\eta_1 + \sigma\delta)(H, H_t) \\ &\quad -\beta\sigma\eta_1(H, Z_t) - ((\sigma+\delta)\eta_2 + \sigma\delta)(E, E_t) - \beta\eta_2(E, (\nabla\times Z)_t) \\ &\quad +\beta(H_t, \nabla(\nabla\cdot Z)) + \beta\eta_1(H, \nabla(\nabla\cdot Z)) - \beta(H_t, Z_{tt}) - \beta\sigma(H_t, Z_t) \\ &\quad -\beta(E_t, (\nabla\times Z)_t) - \|\nabla^3 Z\|_2^2 + (\Delta Z, \Delta(Z\times\Delta Z)) \\ &\quad +(\Delta Z, \Delta(Z\times H)) - k(\Delta Z, \Delta((1+\mu|Z|^2)Z)).\end{aligned} \tag{2.18}$$

However, we have

$$
\begin{aligned}
(H, Z_{tt}) &= (H, Z_t)_t - (H_t, Z_t),\\
(H_t, Z_{tt}) &= (H_t, Z_t)_t - (\Delta H - \beta Z_{tt} - (\delta+\sigma)H_t - \beta\sigma Z_t - \sigma\delta H + \beta\nabla(\nabla\cdot Z), Z_t)\\
&= (H_t, Z_t)_t + \frac{\beta}{2}(Z_t, Z_t)_t - (\Delta H, Z_t) + (\delta+\sigma)(H_t, Z_t) + \beta\sigma\|Z_t\|_2^2\\
&\quad + \sigma\delta(H, Z_t) + \beta(\nabla(\nabla\cdot Z), Z_t),\\
\beta(E_t, (\nabla\times Z)_t) &= \beta(E_t, \nabla\times Z)_t - \beta(\Delta E - (\sigma+\delta)E_t - \delta\sigma E - \beta(\nabla\times Z)_t, \nabla\times Z)\\
&= \beta(E_t, \nabla\times Z)_t + \frac{\beta^2}{2}(\nabla\times Z, \nabla\times Z)_t + \beta(\nabla E, \nabla(\nabla\times Z))\\
&\quad + \beta(\sigma+\delta)(E_t, \nabla\times Z) + \delta\sigma\beta(E, \nabla\times Z),\\
\beta\eta_2(E, (\nabla\times Z)_t) &= \beta\eta_2(E, \nabla\times Z)_t - \beta\eta_2(E_t, \nabla\times Z).
\end{aligned}
$$

Set

$$
e_1(t) = \frac{1}{2}G(t) + R(t), \tag{2.19}
$$

where

$$
\begin{aligned}
G(t) &= \|E_t\|_2^2 + \|H_t\|_2^2 + \|\nabla E\|_2^2 + \|\nabla H\|_2^2 + \|\Delta Z\|_2^2\\
&= 2e(t) - 2\eta_1(H, H_t) - 2\eta_2(E, E_t),
\end{aligned} \tag{2.20}
$$

and

$$
\begin{aligned}
R(t) \;=\; & \beta(H_t, Z_t) + \frac{\beta^2}{2}\|Z_t\|_2^2 + \beta(E_t, \nabla\times Z) + \frac{\beta^2}{2}\|\nabla\times Z\|_2^2\\
& -\eta_2\beta(E, \nabla\times Z) + \frac{1}{2}\sigma\eta_1\|H\|_2^2 + \frac{1}{2}\sigma\eta_2\|E\|_2^2\\
& + \eta_2\beta(E, \nabla\times Z) + \eta_2(E, E_t) + \eta_1(H, H_t).
\end{aligned} \tag{2.21}
$$

Then we have

$$
\begin{aligned}
&\frac{\mathrm{d}e_1(t)}{\mathrm{d}t} + (\sigma-\eta_1)\|H_t\|_2^2 + (\sigma-\eta_2)\|E_t\|_2^2 + \eta_1\delta\|H\|_2^2\\
&+ \eta_1\|\nabla H\|_2^2 + \eta_2\|\nabla E\|_2^2 + \|\nabla\Delta Z\|_2^2 + \sigma\beta^2\|Z_t\|_2^2\\
&= -(2\sigma\beta - \eta_1\beta)(Z_t, H_t) + \beta(Z_t, \Delta H) - (\sigma-\eta_2)\beta(E_t, \nabla\times Z)\\
&\quad - \beta(\nabla E, \nabla(\nabla\times Z)) + (\Delta(Z\times\Delta Z), \Delta Z) + (\Delta(Z\times H), \Delta Z)\\
&\quad - \sigma\beta\eta_1(H, Z_t) + \beta(H_t, \nabla(\nabla\cdot Z)) + \beta\eta_1(H, \nabla(\nabla\cdot Z))\\
&\quad + \beta^2(Z_t, \nabla(\nabla\cdot Z)) - k(\Delta Z, \Delta(1+\mu|Z|^2 Z)).
\end{aligned} \tag{2.22}
$$

In the sequel, we estimate every term of the right hand side of (2.22).

(1) Estimate of the first term

$$
|Z_t|^2 \le |\Delta Z|^2 + |Z\times(\Delta Z + H)|^2 + k^2|(1+\mu|Z|^2)Z|^2.
$$

$$
\begin{aligned}
|-(2\sigma\beta-\eta_1\beta)(Z_t,H_t)| &\le \frac{\sigma-\eta_1}{2}\|H_t\|_2^2+C_1\|Z_t\|_2^2\\
&\le \frac{\sigma-\eta_1}{2}\|H_t\|_2^2+C_2||\Delta Z||_2^2+d_1\\
&\le \frac{\sigma-\eta_1}{2}\|H_t\|_2^2+\frac{1}{4}\|\nabla\Delta Z\|_2^2+d_2. \qquad (2.23)
\end{aligned}
$$

(2) Estimate of the second term. Acting ∇ on (1.1), we get

$$
\nabla Z_t=\nabla\Delta Z+\nabla Z\times(\Delta Z+H)+Z\times(\nabla\Delta Z+\nabla H)-k\nabla[(1+\mu|Z|^2)Z]. \qquad (2.24)
$$

$$
\begin{aligned}
\beta(Z_t,\Delta H) &= -\beta(\nabla Z_t,\nabla H)=-\beta(\Delta\nabla Z,\nabla H)-\beta(\nabla Z\times(\Delta Z+H),\nabla H)\\
&\quad -\beta(Z\times(\nabla\Delta Z),\nabla H)+\beta k(\nabla[(1+\mu|Z|^2)Z],\nabla H)\\
&\le \frac{\beta}{2}(\|\Delta\nabla Z\|_2^2+\|\nabla H\|_2^2)+\beta\|\nabla Z\|_4\|\Delta Z\|_4\|\nabla H\|_2+\beta|(\nabla Z\times H,\nabla H)|\\
&\quad +\beta|(Z\times(\nabla\Delta Z),\nabla H)|+k\beta\|Z\|_\infty^2\|Z\|_2\|\nabla H\|_2\\
&\le \frac{\beta}{2}(\|\Delta\nabla Z\|_2^2+\|\nabla H\|_2^2)+\beta(2\|Z\|_\infty^2\|\nabla^3 Z\|_2\|\nabla H\|_2)\\
&\quad +\beta\|\nabla Z\|_4\|H\|_4\|\nabla H\|_2+\beta\|Z\|_\infty|(\nabla\Delta Z,\nabla H)|\\
&\le \left(\frac{\beta}{2}+2\beta\|Z\|_\infty^2\right)(\|\Delta\nabla Z\|_2^2+\|\nabla H\|_2^2)+\beta\|\nabla Z\|_4\|H\|_4\|\nabla H\|_2\\
&\quad +\beta\|H\|_2\|\nabla H\|_2^{\frac{3}{2}}\|Z\|_\infty\|\nabla^3 Z\|_2^{\frac{1}{4}}\\
&\le \left(\frac{\beta}{2}+2\beta\|Z\|_\infty^2+\beta\|H\|_2\|Z\|_\infty\right)(\|\nabla^3 Z\|_2+\|\nabla H\|_2^2)\\
&\le C_2(\|\nabla^3 Z\|_2+\|\nabla H\|_2^2). \qquad (2.25)
\end{aligned}
$$

(3) Estimate of the third term

$$
|(\sigma+\eta_2)\beta(E_t,\nabla\times Z)|\le\varepsilon_2\|E_t\|_2^2+C_1\|\nabla Z\|_2^2\le\varepsilon_2\|E_t\|_2^2+C_2. \qquad (2.26)
$$

(4) Estimate of the forth term

$$
|-\beta(\nabla E,\nabla(\nabla\times Z))|\le\varepsilon_3\|\nabla\Delta Z\|_2^2+C_1\|E\|_2^2\le\varepsilon_3\|\nabla\Delta Z\|_2^2+C_2. \qquad (2.27)
$$

(5) Estimate the fifth term

$$
|(\Delta(Z\times\Delta Z),\Delta Z)|\le|(\nabla(Z\times\Delta Z),\nabla\Delta Z)|\le\varepsilon_4\|\nabla\Delta Z\|_2^2+\|\Delta Z\|_2^4. \qquad (2.28)
$$

(6) Estimate the sixth term

$$
\begin{aligned}
|(\Delta(Z\times H),\Delta Z)| &= |(\nabla(Z\times H),\nabla^3 Z)|\\
&\le \|\nabla Z\|\infty\|H\|_2\|\nabla\Delta Z\|_2+\|\nabla H\|_2\|\nabla\Delta Z\|_2 \qquad (2.29)\\
&\le \varepsilon_5\|\nabla\Delta Z\|_2^2+\frac{1}{2}(\|\nabla H\|_2^2+\|\nabla\Delta Z\|_2^2).
\end{aligned}
$$

(7) Estimate of the seventh term

$$
|-\sigma\beta\eta_1(H,Z_t)|\le c_1\|H\|_2\|Z_t\|_2\le c_2\|\Delta Z\|_2+d_1\le\varepsilon_6\|\nabla\Delta Z\|_2^2+d_2. \qquad (2.30)
$$

(8) Estimate of the eighth term

$$|\beta(H_t, \nabla(\nabla \cdot Z))| \leq \beta \|H_t\| \|\Delta Z\|_2 \leq \varepsilon_7 \|H_t\|_2^2 + \varepsilon_8 \|\nabla \Delta Z\|_2^2 + C_1. \tag{2.31}$$

(9) Estimate of the ninth term

$$|\beta\eta_1(H, \nabla(\nabla Z))| \leq \eta_1 \beta \|H\|_2 \|\Delta Z\|_2 \leq \varepsilon_8 \|\nabla \Delta Z\|_2^2 + C_1. \tag{2.32}$$

(10) Estimate of the tenth term

$$|\beta^2(Z_t, \nabla(\nabla \cdot Z))| \leq \beta^2 \|\Delta Z\|_2^2 + C_1 \leq \varepsilon_9 \|\nabla \Delta Z\|_2^2 + C_2. \tag{2.33}$$

Choosing β suitable small, there exists a constant $a > 0$ such that

$$\frac{\mathrm{d}e_1(t)}{\mathrm{d}t} + a(\|H_t\|_2^2 + \|E_t\|_2^2 + \|\nabla H\|_2^2 + \|\nabla E\|_2^2 + \|H\|_2^2 + \|\nabla \Delta Z\|_2^2) \leq C, \tag{2.34}$$

where C is independent of t. Since $\Delta Z(x,t)$ is periodic with respect to x, $\int_0^{2D} \Delta Z \mathrm{d}x = 0$. By Poincaré inequality, we have

$$\|\Delta Z\|_2^2 \leq \delta \|\nabla \Delta Z\|_2^2. \tag{2.35}$$

Choosing $\delta_0 = \min(a, a/\delta_0)$, we have from (2.34) that

$$\frac{\mathrm{d}e_1(t)}{\mathrm{d}t} + 2\delta_0 e_1(t) \leq C + 2\delta_0 R(t) \leq C + 2\delta_0 \sup_t R(t). \tag{2.36}$$

Hence, we have

$$e_1(t)\mathrm{e}^{2\delta_0 t} \leq e_1(0) + \left(\frac{C}{2\delta_0} + \sup_t R(t)\right)(\mathrm{e}^{2\delta_0} - 1). \tag{2.37}$$

Moreover,

$$e_1(t) \leq e_1(0) + \frac{C}{2\delta_0} + \sup_t |R(t)|, \tag{2.38}$$

that is,

$$G(t) \leq 2C_0 + 2(\sup_t |R(t)| - R(t)) \leq 2C_0 + 4 \sup_t |R(t)|. \tag{2.39}$$

it follows from (2.21) that

$$\begin{aligned}
R(t) \leq\; & \beta \|Z_t\|_2 \|H_t\|_2 + \frac{\beta^2}{2} \|Z_t\|_2^2 + \beta \|E_t\|_2 \|\nabla Z\|_2 + \eta_1 \|H\|_2 \|\nabla E\|_2 \\
& + \eta_2 \|E\|_2 \|E_t\|_2 + \frac{\beta^2}{2} \|\nabla Z\|_2^2 \\
& + \frac{1}{2}\sigma\eta_1 \|H\|_2^2 + \frac{1}{2}\sigma\eta_2 \|E\|_2^2 + \beta\eta_2 \|E\|_2 \|\nabla Z\|_2 \\
\leq\; & \frac{\beta + \beta^2}{2} \|Z_t\|_2^2 + \frac{\beta}{2}(\|H_t\|_2^2 + \|E_t\|_2^2 + \|\nabla E\|_2^2) + C_1 \\
\leq\; & (\beta + \beta^2) \|\Delta Z\|_2^2 + \frac{\beta}{2}(\|H_t\|_2^2 + \|E_t\|_2^2 + \|\nabla E\|_2^2) + C_2.
\end{aligned} \tag{2.40}$$

Taking $\beta < \frac{1}{2}$ and $\beta + \beta^2 < \frac{1}{4}$, we have

$$a_0 = 4\max\{\frac{\beta}{2}, (\beta + \beta^2)\} < 1. \tag{2.41}$$

It follows from (2.40) that

$$|R(t)| \le \frac{1}{4}a_0(\|H_t\|_2^2 + \|E_t\|_2^2 + \|\nabla E\|_2^2 + \|\Delta Z\|_2^2) \le \frac{1}{4}a_0 G(t). \tag{2.42}$$

Substituting (2.42) into (2.39), we have

$$G(t) \le 2c_0 + a_0 \sup_t G(t), \tag{2.43}$$

that is,

$$\sup_t G(t) \le \frac{2c_0}{1-a_0} = d_0. \tag{2.44}$$

We can prove the following existence theorem by using the above estimates.

Lemma 2.4 *Assume that $Z(x,t)$, $H(x,t)$, $E(x,t)$ are smooth solutions of* (1.1)-(1.6), $(Z_0(x), H_0(x), E_0(x)) \in (H^2(\Omega), H^1(\Omega), H^1(\Omega))$, $\Omega \subset \mathbb{R}^d, 1 \le d \le 2$ *and satisfying the following conditions*
(i) $\eta_1 > 0,\ \eta_2 > 0,\ \sigma > \eta_1 + \eta_2 + 1$,
(ii) $0 < \beta < \frac{1}{2},\ \beta + \beta^2 < \frac{1}{4}$,
(iii) When $d = 2$, $\|Z_0(x)\|_{H^2}^2 \le \lambda$ *with* $\lambda = \lambda(\beta)$ *is a small constant, we have*

$$\sup_t[\|Z(\cdot,t)\|_{H^2(\Omega)}^2 + \|H(\cdot,t)\|_{H^1(\Omega)}^2 + \|E(\cdot,t)\|_{H^1(\Omega)}^2] \le K, \tag{2.45}$$

where the K is a constant only depends on $\|Z_0(x)\|_{H^2(\Omega)}^2, \|H_0(x)\|_{H^1(\Omega)}^2, \|E_0(x)\|_{H^1(\Omega)}^2$, *and is independent of t.*

Next the Theorem 1.2 can be obtained by using the induction method.

3 Global attractors and the Hausdorff dimension and fractal dimension

From Theorem 1.2 we obtain that problem (1.1)-(1.6) has a semigroup operator $S(t)(Z_0, H_0, E_0)$ which is continuous by [13, 14]. Taking the subset D such that

$$D = \{(Z, E, H) \in H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega), \nabla\cdot E = 0, \nabla\cdot(H + \beta Z) = 0\},$$

it follows from Lemma 2.2 that the operator $S(t) : D \to D$ is bounded and

$$\begin{aligned}\bar{A} = \{&\nabla\cdot E = 0, \nabla\cdot(H+\beta Z) = 0, Z(\cdot,t) \in H^2(\Omega), H(\cdot,t) \in H^1(\Omega), E(\cdot,t) \in H^1(\Omega),\\ &\|Z(\cdot,t)\|_{H^2(\Omega)}^2 + \|H(\cdot,t)\|_{H^1(\Omega)}^2 + \|E(\cdot,t)\|_{H^1(\Omega)}^2 \le K\}\end{aligned}$$

is a bounded absorbing set in D, then we get $\mathcal{A} = \omega(\bar{A})$ is a weakly compact attractor of the periodic initial value problem (1.1)-(1.6). This completes the proof of the Theorem 1.3.

Lemma 3.1 *The smooth solution* $(Z(x,t),H(x,t),E(x,t))$ *of problem* (1.1)-(1.6) *is continuously dependent on the initial data.*

Proof. Let $(Z_i(x,t),H_i(x,t),E_i(x,t))$ $(i=1,2)$ be smooth solution of (1.1)-(1.6) with initial conditions $Z_i(x,0)=Z_{0i}(x)$, $H_i(x,0)=H_{0i}(x)$, $E_i(x,0)=E_{0i}(x)$ $(i=1,2)$. Let

$$\begin{aligned} Z(x,t) &= Z_2(x,t)-Z_1(x,t),\\ H(x,t) &= H_2(x,t)-H_1(x,t),\\ E(x,t) &= E_2(x,t)-E_1(x,t), \end{aligned}$$

then we get that $(Z(x,t),H(x,t),E(x,t))$ satisfies

$$\begin{aligned} & Z_t = \Delta Z + Z\times\Delta Z_2 + Z_1\times\Delta Z + Z\times H_2 + Z_1\times H\\ &\qquad - k(Z+\mu|Z_1|^2Z+\mu(|Z_2|^2-|Z_1|^2)Z_2), && (3.1)\\ & E_t = \nabla\times H-\sigma E, && (3.2)\\ & H_t = -\nabla\times E-\beta Z_t-\delta H, && (3.3)\\ & \nabla\cdot(H+\beta Z)=0, \quad \nabla\cdot E=0.\\ & Z(x+D,t)=Z(x-D,t), \quad H(x+D,t)=H(x-D,t),\\ & E(x+D,t)=E(x-D,t), && (3.4)\\ & Z(x,0)=Z_0(x), H(x,0)=H_0(x), E(x,0)=E_0(x).\\ & \nabla\cdot(H_0+\beta Z_0)=0, \quad \nabla\cdot E_0=0. && (3.5) \end{aligned}$$

We may establish inequality as follows

$$\sup_{0\le t\le T}[\|\nabla Z(\cdot,t)\|_2^2+\|H(\cdot,t)\|_2^2+\|E(\cdot,t)\|_2^2]\le C[\|\nabla Z_0(x)\|_2^2+\|H_0(x)\|_2^2+\|E_0(x)\|_2^2], \quad (3.6)$$

where C is an absolute constant. It is clear if (3.6) holds, then the conclusion of Lemma 3.1 is proved.

In fact, taking the inner product of (3.1) with ΔZ, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla Z|^2\mathrm{d}x+\|\Delta Z\|_2^2\\ &= -\int_\Omega Z\times\Delta Z_2\cdot\Delta Z\mathrm{d}x-\int_\Omega Z\times H_2\cdot\Delta Z\mathrm{d}x-\int_\Omega Z_1\times H\cdot\Delta Z\mathrm{d}x\\ &+k\int_\Omega Z\cdot\Delta Z\mathrm{d}x+k\mu\int_\Omega|Z_1|^2Z\cdot\Delta Z\mathrm{d}x+k\mu\int_\Omega(|Z_2|^2-|Z_1|^2)Z_2\cdot\Delta Z\mathrm{d}x, \end{aligned}$$

where

$$\begin{aligned} |-\int_\Omega Z\times\Delta Z_2\cdot\Delta Z\mathrm{d}x| &= |\int_\Omega Z\times\nabla\Delta Z_2\cdot\nabla Z\mathrm{d}x|\le||\nabla\Delta Z||_\infty||Z||_2||\nabla Z||_2\\ &\le C_1(||Z||_2^2+||\nabla Z||_2^2),\\ |-\int_\Omega Z\times H_2\cdot\Delta Z\mathrm{d}x| &= |\int_\Omega Z\times\nabla H_2\cdot\nabla Z\mathrm{d}x|\le||\nabla H_2||_\infty||Z||_2||\nabla Z||_2\\ &\le C_2(||Z||_2^2+||\nabla Z||_2^2),\\ |k\mu\int_\Omega|Z_1|^2Z\cdot\Delta Z\mathrm{d}x| &\le ||Z_1||_\infty^2||Z||_2||\Delta Z||_2\le C_3(||Z||_2^2+||\Delta Z||_2^2), \end{aligned}$$

then we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla Z|^2\mathrm{d}x + \|\Delta Z\|_2^2 + \int_\Omega Z_1\times H\cdot\Delta Z\mathrm{d}x \le C_4(||Z||_2^2 + ||\Delta Z||_2^2 + ||\nabla Z||_2^2). \quad (3.7)$$

Multiplying (3.2) by E and multiplying (3.3) by H and integrating with respect to x over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |E|^2\mathrm{d}x = \int_\Omega \nabla\times H\cdot E\mathrm{d}x - \sigma\int_\Omega |E|^2\mathrm{d}x,\\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |H|^2\mathrm{d}x = -\int_\Omega \nabla\times E\cdot H\mathrm{d}x - \beta\int_\Omega Z_t\cdot H\mathrm{d}x - \delta\int_\Omega |H|^2\mathrm{d}x,
\end{aligned}$$

By using the formula

$$(\nabla\times H)\cdot E - (\nabla\times E)\cdot H = \nabla\cdot(H\times E),$$

we obtain

$$\frac{1}{2\beta}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|E|^2+|H|^2)\mathrm{d}x + \frac{\sigma}{\beta}||E||_2^2 + \frac{\delta}{\beta}||H||_2^2 = -\int_\Omega Z_t\cdot H\mathrm{d}x. \quad (3.8)$$

From (3.7) and (3.8), we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\nabla Z|^2 + \frac{1}{\beta}(|E|^2+|H|^2))\mathrm{d}x + ||\Delta Z||_2^2 + \frac{1}{\beta}(\sigma||E||_2^2 + \delta||H||_2^2)\\
&\le -\int_\Omega (Z_t\cdot H + Z_1\times H\cdot\Delta Z)\mathrm{d}x + C_5(||Z||_2^2 + ||\Delta Z||_2^2 + ||\nabla Z||_2^2).
\end{aligned}$$

Taking the inner product of (3.1) with H, we have

$$\begin{aligned}
\int_\Omega (Z_t\cdot H + Z_1\times H\cdot\Delta Z)\mathrm{d}x \ &\le |\int_\Omega (Z\times\Delta Z_2 + Z\times H_2)\cdot H\mathrm{d}x| + |\int_\Omega \Delta Z\cdot H\mathrm{d}x|\\
&+|k\int_\Omega Z\cdot H\mathrm{d}x| + |k\mu\int_\Omega |Z_1|^2 Z\cdot H\mathrm{d}x|\\
&+|k\mu\int_\Omega (|Z_2|^2 - |Z_1|^2)Z_2\cdot H\mathrm{d}x|\\
&\le C_6(||Z||_2^2 + ||\Delta Z||_2^2 + ||\nabla Z||_2^2 + ||H||_2^2).
\end{aligned}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\nabla Z|^2 + \frac{1}{\beta}(|E|^2+|H|^2))\mathrm{d}x \le C_7(||Z||_2^2 + ||\nabla Z||_2^2 + ||H||_2^2),$$

and the lemma is proved.

In order to prove that the operator semigroup $S(t)$ is Frechet differentiable, we consider a linear variational problem of (1.1)-(1.6) as follows

$$w_t = \Delta w + w\times(\Delta Z_1 + H_1) + Z_1\times(\Delta w + I) - k(1+\mu|Z_1|^2)w, \quad (3.9)$$

$$\nabla\times I = F_t + \sigma F, \quad (3.10)$$

$$\nabla\times F = -I_t - \beta w_t + \delta I, \quad (3.11)$$

$$\nabla\cdot(H_1 + \beta E_1) = 0, \quad \nabla\cdot E_1 = 0, \quad (3.12)$$

$$(w(t), I(t), F(t))|_{t=0} = (Z_0, H_0, E_0), \quad (3.13)$$

where $(Z_1, H_1, E_1) = S(t)(Z_{01}, H_{01}, E_{01})$ is a solution of (1.1)-(1.6) with initial data (Z_{01}, H_{01}, E_{01}). Set

$$
\begin{aligned}
(\tilde{Z}, \tilde{H}, \tilde{E}) = (Z, H, E) - (w, I, F) = S(t)(Z_{01}, H_{01}, E_{01}) \\
-S(t)(Z_0, H_0, E_0) - DS(t)(Z_{01}, H_{01}, E_{01})(Z_0, H_0, E_0).
\end{aligned} \tag{3.14}
$$

Hence

$$
\begin{aligned}
&\tilde{Z}_t = \tilde{Z} \times (\Delta Z_2 + H_2) - w(\Delta Z_1 + H_1) + \Delta Z_1 \\
&\qquad + Z_1 \times (\Delta Z + H) - Z \times (\Delta w + I), && (3.15) \\
&\nabla \times \tilde{H} = \tilde{E}_t + \sigma \tilde{E}, \\
&\nabla \times \tilde{E} = -\tilde{H}_t - \beta \tilde{Z}_t - \delta \tilde{H} && (3.16) \\
&\nabla \cdot (\tilde{H} + \beta \tilde{Z}) = 0, \quad \nabla \cdot \tilde{E} = 0, && (3.17) \\
&(\tilde{Z}, \tilde{H}, \tilde{E})|_{t=0} = 0. && (3.18)
\end{aligned}
$$

Rewrite (3.15) as follows

$$
\tilde{Z}_t = \tilde{Z} \times (\Delta Z_1 + H_1) + Z \times (\Delta Z + H) + \tilde{Z}_1 \times (\Delta \tilde{Z} + \tilde{H}) + \Delta Z_1. \tag{3.19}
$$

It follows from (3.19) that

$$
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{Z}\|_2^2 + \|\nabla Z_1\|_2^2 \leq c_1 \|\tilde{Z}\|_2^2 + c_2 (\|Z\|_2^2 + \|H\|_2^2 + \|E\|_2^2)^2. \tag{3.20}
$$

This implies

$$
\begin{aligned}
\|\tilde{Z}(t)\|_2^2 &\leq \|\tilde{Z}(0)\|_2^2 \mathrm{e}^{c_1 t} + \int_0^t \mathrm{e}^{c_1 (t-\tau)} c_2 (\|Z(s)\|_2^2 + \|H(s)\|_2^2 + \|E(s)\|_2^2)^2 \mathrm{d}s \\
&\leq \int_0^t \mathrm{e}^{c_1 (t-\tau)} c_3 (\|Z(s)\|_2 + \|H(s)\|_2 + \|E(s)\|_2)^4 \mathrm{d}s.
\end{aligned} \tag{3.21}
$$

It follows from Lemma 3.1 that for $0 \leq t \leq T$

$$
\|\tilde{Z}(t)\|_2 \leq C (\|Z_0\|_2^2 + \|H_0\|_2^2 + \|E_0\|_2^2)^2.
$$

Similarly, we can estimate $\|\tilde{H}(t)\|_2$, $\|\tilde{E}(t)\|_2$. And we give the following lemma.

Lemma 3.2 *If solution of problem* (1.1)-(1.6) *are properly smooth, then* $S(t) : (Z_0, H_0, E_0) \to (Z(t), H(t), E(t))$ *is uniformly differentiable. Its differential at* (Z_0, H_0, E_0) *belong to* $\mathcal{A}$ *and*

$$
DS(t)(Z_0, H_0, E_0) = (w(t), I(t), F(t)) \tag{3.22}
$$

is a solution of (3.9)-(3.13).

Now we estimate the Hausdorff dimension and fractal dimension of $\mathcal{A}$. To this aim we consider the linear variational problem of problem (1.1)-(1.6)

$$
\begin{aligned}
z_t &= \Delta z + Z \times \Delta z + Z \times h - (\Delta Z + H) \times z - k(1 + \mu |Z|^2) z, && (3.23) \\
e_t &= \nabla \times h - \sigma e, && (3.24) \\
h_t &= -\nabla \times e - \beta z_t + \delta h, && (3.25) \\
z(0) &= z_0, \ h(0) = h_0, \ e(0) = e_0. && (3.26)
\end{aligned}
$$

Simply write (3.23)-(3.26) as operator form

$$v_t = -L(u)v,\ v(0) = v_0, \tag{3.27}$$

where $v = (z, e, h)$, $u = (Z, E, H)$, $v_0 = (z_0, e_0, h_0)$, z, e, h, Z, E, H are all three dimensional vector valued functions.

Since the periodic initial value problem (1.1)-(1.6) has smooth solution, the coefficients of linear system (3.23)-(3.26) admits global smooth solution. Denote its solution operator by $G(t)$, that is, $v(t) = G(t)v_0$. Moreover, we can prove that the semigroup operator $S(t)u_0$ is differential in $L^2(\Omega)$ and the Frechet differential $S'(t)u_0 = G(t)v_0$.

Now we estimate the Hausdorff dimension and fractal dimension of $\mathcal{A}$. Rewrite (3.23)-(3.25) as

$$z_t + f(z, \nabla z, \Delta z, h; Z, \nabla Z, \Delta Z, H) = 0, \tag{3.28}$$

$$e_t + \sigma e - \nabla \times h = 0, \tag{3.29}$$

$$h_t + \beta z_t + \nabla \times e = \delta h. \tag{3.30}$$

where

$$\begin{aligned} &f(z, \nabla z, \Delta z, h; Z, \nabla Z, \Delta Z, H) \\ &= -\Delta z - Z \times \Delta z - Z \times h + (\Delta Z + H) \times z + k(1 + \mu|Z|^2)z. \end{aligned} \tag{3.31}$$

Choosing periodic orthogonal function basis $(\varphi_j(x), e_j(x), h_j(x))$ such that
(i) $\Delta\varphi_j = -\lambda_j^2\varphi_j$,
(ii) $\|\varphi_j\|_2 = \|e_j\|_2 = \|h_j\|_2 = 1$. We have

$$\|\nabla\varphi_j\|_2 = |\lambda_j|,\ \|\Delta\varphi_j\|_2 = \lambda_j^2.$$

From the definition, we have

$$\begin{aligned} \mathrm{Trac}[L(u(t)) \cdot Q_J(t)] = \sum_{j=1}^{J} \Bigg[&\Big(f(\varphi_j, \nabla\varphi_j, \Delta\varphi_j, h_j; Z, \nabla Z, \Delta Z, H), \varphi_j \Big) \\ &-(\nabla \times h_j, e_j) + (\nabla \times e_j, h_j) - \delta(h_j, h_j) \\ &-\beta\Big(f(\varphi_j, \nabla\varphi_j, \Delta\varphi_j, h_j; Z, \nabla Z, \Delta Z, H), h_j \Big) \Bigg] \\ &+\sigma(e_j, e_j). \end{aligned} \tag{3.32}$$

Since

$$-(\nabla \times h_j, e_j) + (\nabla \times e_j, h_j) = \int_\Omega \nabla \cdot (e_j \times h_j)\mathrm{d}x = 0,$$

we only need to estimate the following two terms in (3.32)

$$\Big(f(\varphi_j, \nabla\varphi_j, \Delta\varphi_j, h_j; Z, \nabla Z, \Delta Z, H), \varphi_j \Big)$$

and

$$\Big(f(\varphi_j, \nabla\varphi_j, \Delta\varphi_j, h_j; Z, \nabla Z, \Delta Z, H), h_j\Big).$$

From (3.31) we have

$$\begin{aligned}\Big(f(\varphi_j, \nabla\varphi_j, \Delta\varphi_j, h_j; Z, \nabla Z, \Delta Z, H), \varphi_j\Big) = \ & -(\Delta\varphi_j, \varphi_j) - (Z \times \Delta\varphi_j, \varphi_j) \\ & -(Z \times h_j, \varphi_j) + ((\Delta Z + H) \times \varphi_j, \varphi_j) \\ & +(k(1+\mu|Z|^2)\varphi_j, \varphi_j)\end{aligned}$$

in which

$$\begin{aligned}&-(\Delta\varphi_j, \varphi_j) = \lambda_j^2, \\ &|(Z \times \Delta\varphi_j, \varphi_j)| = |(\nabla\varphi_j, \nabla Z \times \varphi_j)| \le \|\nabla\varphi_j\|_2 \|\varphi_j\|_2 \|\nabla Z\|_\infty = |\lambda_j| \|\nabla Z\|_\infty, \\ &|(Z \times h_j, \varphi_j)| \le \|h_j\|_2 \|\varphi_j\|_2 \|Z\|_\infty = \|Z\|_\infty \le \|Z_0\|_{H^2}, \\ &|((\Delta Z + H) \times \varphi_j, \varphi_j)| \le |(\Delta Z \times \varphi_j, \varphi_j)| + |(H \times \varphi_j, \varphi_j)| \\ &\qquad\qquad\qquad\qquad\qquad \le \|\nabla\varphi_j\|_2 \|\varphi_j\|_2 \|\nabla Z\|_\infty + ||H||_\infty ||\varphi_j||_2^2 \\ &\qquad\qquad\qquad\qquad\qquad = |\lambda_j| ||\nabla Z||_\infty + ||H||_\infty \\ &|(k(1+\mu|Z|^2)\varphi_j, \varphi_j)| \le k||\varphi_j||_2^2 + k\mu||Z||_\infty^2 ||\varphi_j||_2^2 = k + k\mu||Z||_\infty^2\end{aligned}$$

Hence,

$$\Big(f(\varphi_j, \nabla\varphi_j, \Delta\varphi_j, h_j; Z, \nabla Z, \Delta Z, H), \varphi_j\Big) \ge \lambda_j^2 - |\lambda_j| \|\nabla Z\|_\infty - \|Z_0\|_{H^2}. \tag{3.33}$$

Similarly, it follows

$$\begin{aligned}-\beta\Big(f(\varphi_j, \nabla\varphi_j, \Delta\varphi_j, h_j; Z, \nabla Z, \Delta Z, H), h_j\Big) = \ & \beta(\Delta\varphi_j, h_j) + \beta(Z \times \Delta\varphi_j, h_j) \\ & -\beta((\Delta Z + H) \times \varphi_j, h_j) \\ & -\beta(k(1+\mu|Z|^2)\varphi_j, h_j)\end{aligned}$$

in which

$$\begin{aligned}|-\beta((\Delta Z + H) \times \varphi_j, h_j)| &\le \beta\|\Delta Z + H\|_\infty \|\varphi_j\|_2 \|h_j\|_2 = \beta\|\Delta Z + H\|_\infty, \\ |\beta(\Delta\varphi_j, h_j)| &\le \beta\|\Delta\varphi_j\|_2 \|h_j\|_2 = \beta\lambda_j^2, \\ |\beta(Z \times \Delta\varphi_j, h_j)| &\le \beta\lambda_j^2 \|Z\|_\infty \le \beta\lambda_j^2 \|Z_0\|_{H^2}.\end{aligned}$$

Hence,

$$\begin{aligned}&-\beta\Big(f(\varphi_j, \nabla\varphi_j, \Delta\varphi_j, h_j; Z, \nabla Z, \Delta Z, H), h_j\Big) \\ &\ge -\beta\lambda_j^2 - \beta\|Z_0\|_{H^2}\lambda_j^2 - \beta\|\Delta Z + H\|_\infty.\end{aligned} \tag{3.34}$$

On the other hand, we have

$$
\begin{aligned}
\sigma(e_j, e_j) &= \sigma, \\
\delta(e_j, e_j) &= \delta, \\
|(\nabla \times h_j, e_j)| &\le \|\nabla h_j\|_2 \|e_j\|_2 = |\lambda_j|, \\
|(\nabla \times e_j, h_j)| &\le \|\nabla e_j\|_2 \|h_j\|_2 = |\lambda_j|.
\end{aligned} \tag{3.35}
$$

Substituting (3.33), (3.34) and (3.35) into (3.32), we have

$$
\begin{aligned}
\mathrm{Trac}[L(u(t)) \cdot Q_J(t)] \ge & \sum_{j=1}^{J} \Big(1 - \beta(1 + \|Z_0\|_{H^2}) \Big) \lambda_j^2 + \sum_{j=1}^{J} (2 - ||\nabla Z||_\infty) |\lambda_j| \\
& + (-\beta \|\Delta Z + H\|_\infty + \sigma - \delta - ||Z_0||_{H_2}) J.
\end{aligned} \tag{3.36}
$$

Let β be a small value such that $\beta(1 + ||Z_0||_{H_2}) < 1$,
setting

$$
\begin{aligned}
& \delta_1 = 1 - \beta(1 + \|Z_0\|_{H^2}), \ X = \Big(\sum_{j=1}^{J} \lambda_j^2 \Big)^{\frac{1}{2}}, \\
& a = 2 - ||\nabla Z||_\infty, \ b = -\beta \|\Delta Z + H\|_\infty + \sigma - \delta - ||Z_0||_{H_2}.
\end{aligned}
$$

Noting that

$$
\sum_{j=1}^{J} |\lambda_j| \le \Big(\sum_{j=1}^{J} \lambda_j^2 \Big)^{\frac{1}{2}} J^{\frac{1}{2}},
$$

(3.36) can be rewrote as

$$
\mathrm{Trac}[L(u(t)) \cdot Q_J(t)] \ge \delta_1 X^2 - a J^{\frac{1}{2}} X + bJ = \delta_1 \Big(X - \frac{J^{\frac{1}{2}}}{\delta_1} \Big)^2 + \frac{4b\delta_1 - a^2}{4\delta_1} J. \tag{3.37}
$$

when $4b\delta_1 - a^2 < 0$ we have

$$
\mathrm{Trac}[L(u(t)) \cdot Q_J(t)] \ge \delta_1 \Big(X - \frac{a + \sqrt{a^2 - 4\delta_1 b}}{2\delta_1} J^{\frac{1}{2}} \Big) \Big(X - \frac{a - \sqrt{a^2 - 4\delta_1 b}}{2\delta_1} J^{\frac{1}{2}} \Big).
$$

Let J satisfy

$$
X \ge \frac{a + \sqrt{a^2 - 4\delta_1 b}}{2\delta_1} J^{\frac{1}{2}}, \tag{3.38}
$$

and estimate λ_j as in [15] as follows

$$
\lambda_j^2 \ge \Big[\frac{(j-1)^{\frac{1}{d}}}{2} - 1 \Big]^2 = \frac{1}{4} (j-1)^{\frac{2}{d}} - (j-1)^{\frac{1}{d}} + 1,
$$

that is

$$
\lambda_j^2 \ge \begin{cases} \frac{1}{4}(j-1)^2 - j + 2 = \frac{1}{4} j^2 - \frac{3}{2} j + \frac{9}{4}, & d = 1, \\ \frac{1}{4}(j-1) - (j-1)^{\frac{1}{2}} + 1 = \frac{1}{4} j + \frac{3}{4} - (j-1)^{\frac{1}{2}}, & d = 2. \end{cases} \tag{3.39}
$$

(i) $d=1$

$$\begin{aligned}\sum_{j=1}^{J}\lambda_j^2 &\geq \frac{1}{4}\sum_{j=1}^{J}j^2-\frac{3}{2}\sum_{j=1}^{J}j+\frac{9}{4}J\\ &=\frac{1}{24}(J+1)(2J+1)J-\frac{3}{4}(J+1)J+\frac{9}{4}J\\ &=\frac{1}{12}J^3-\frac{8}{5}J^2+\frac{37}{24}J.\end{aligned}$$

In order to (3.38), choose J_0 such that

$$\frac{1}{12}J_0^3-\frac{8}{5}J_0^2+\left[\frac{37}{24}-\frac{a+\sqrt{a^2-4\delta_1 b}}{4\delta_1^2}\right]J_0>0, \tag{3.40}$$

that is,

$$\begin{aligned}&2J_0^2-15J_0+37-\frac{6(a+\sqrt{a^2-4\delta_1 b})^2}{\delta_1^2}>0,\\ &(J_0-\frac{15}{4})^2+\frac{71}{16}-\frac{3(a+\sqrt{a^2-4\delta_1 b})^2}{\delta_1^2}>0.\end{aligned}$$

When

$$(J_0-\frac{15}{4})^2+\frac{71}{16}-\frac{3(a+\sqrt{a^2-4\delta_1 b})^2}{\delta_1^2}>0,$$

that is

$$(a+\sqrt{a^2-4\delta_1 b})^2<2\delta_1, \tag{3.41}$$

we may choose $J_0=1$.

When

$$(a+\sqrt{a^2-4\delta_1 b})^2\geq 2\delta_1,$$

we may choose

$$J_0>\sqrt{\frac{3(a+\sqrt{a^2-4\delta_1 b})^2}{\delta_1^2}-\frac{71}{16}}+\frac{15}{4}. \tag{3.42}$$

(ii) $d=2$

$$\begin{aligned}\sum_{j=1}^{J}\lambda_j^2 &\geq \frac{1}{4}\sum_{j=1}^{J}j+\frac{3}{4}J-\sum_{j=1}^{J}(j-1)^{\frac{1}{2}}\\ &=\frac{1}{8}(J+1)J+\frac{3}{4}J-\sum_{j=1}^{J-1}j^{\frac{1}{2}}\\ &\geq\frac{J^2+7J}{8}-(\sum_{j=1}^{J-1}j)^{\frac{1}{2}}\sqrt{J-1}\\ &\geq\frac{J^2+75}{8}-\frac{\sqrt{2}}{2}J^{\frac{1}{2}}.\end{aligned}$$

In order to (3.38), choose J_0 such that

$$\frac{J_0+7}{8}-\frac{\sqrt{2}}{2}J_0^{\frac{1}{2}}>\frac{(a+\sqrt{a^2-4\delta_1 b})^2}{4\delta_1^2}, \tag{3.43}$$

that is,

$$(J_0^{\frac{1}{2}}-2\sqrt{2})^2-\left(1+2\left(\frac{a+\sqrt{a^2-4\delta_1 b}}{\delta_1}\right)^2\right)>0,$$

$$J_0>\left\{2\sqrt{2}+\left[1+\left(\frac{a+\sqrt{a^2-4\delta_1 b}}{\delta_1}\right)^2\right]^{\frac{1}{2}}\right\}^2>0.$$

We have from the above results the following theorem.

Theorem 3.1 *Let $\Omega\subset\mathbb{R}^d$ $(1\le d\le 2)$ be bounded set and assume*
(i) $\sigma>\beta\|\Delta Z+H\|_\infty+\delta+||Z_0||_{H^2}$,
(ii) $0<\beta<\frac{1}{1+\|Z_0\|_{H^2}}$, $\beta<\frac{1}{2}$, $\beta+\beta^2<\frac{1}{4}$,
(iii) when $d=2$, $\|Z_0\|_{H^2}<<1$.

Then there exist an attractor $\mathcal{A}=\omega(\bar{A})$ of the periodic initial value problem (1.1)-(1.6) *with*

$$\bar{A}=\{(Z,H,E)\in(H^2(\Omega),H^1(\Omega),H^1(\Omega)),\|Z\|_{H^2}+\|H\|_{H^1}+\|E\|_{H^1}\le K\} \tag{3.44}$$

is a bounded absorbing set. The Hausdorff dimension and Fractal dimension of $\mathcal{A}$ are finite and satisfy
(1) if $a^2-4\delta_1 b<0$, *then*

$$d_H(\mathcal{A})\le 1,\quad d_F(\mathcal{A})\le 2, \tag{3.45}$$

(2) if $a^2-4\delta_1 b>0$,
(a) when $d=1$ *and* $a+\sqrt{a^2-4\delta_1 b}<2\delta_1$, *then*

$$d_H(\mathcal{A})\le 1,\quad d_F(\mathcal{A})\le 2, \tag{3.46}$$

when $d=1$ *and* $a+\sqrt{a^2-4\delta_1 b}>2\delta_1$, *then*

$$d_H(\mathcal{A})\le J_1,\quad d_F(\mathcal{A})\le 2J_1, \tag{3.47}$$

where J_1 is the smallest integer satisfies

$$J_1>\sqrt{\frac{3(a+\sqrt{a^2-4\delta_1 b})^2}{\delta_1^2}-\frac{71}{16}+\frac{15}{4}},$$

(b) when $d=2$, *then*

$$d_H(\mathcal{A})\le J_2,\quad d_F(\mathcal{A})\le 2J_2, \tag{3.48}$$

where J_2 is the smallest integer satisfies

$$J_2>\left\{2\sqrt{2}+\left[1+\left(\frac{a+\sqrt{a^2-4\delta_1 b}}{\delta_1}\right)^2\right]^{\frac{1}{2}}\right\}^2,$$

in which

$$\delta_1=1-\beta(1+\|Z_0\|_{H^2}),\quad a=2-||\nabla Z||_\infty,\quad b=\sigma-\beta\|\Delta Z+H\|_\infty-\delta-||Z_0||_{H^2}.$$

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