# Global existence and nonexistence of solutions to the Cahn-Hilliard equation with variable exponent sources


Quach Van Chuong[a], Le Cong Nhan[b], and Le Xuan Truong[c]

[a] *Department of Mathematics, Dong Nai University, Bien Hoa City, Dong Nai Province, Vietnam*
[b] *Faculty of Applied Sciences, HCMC University of Technology and Education, Ho Chi Minh City, Vietnam*
[c] *Department of Mathematics and Statistics, University of Economics Ho Chi Minh City, Vietnam*


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**Abstract**

This paper deals with a Cahn-Hilliard equation with variable exponent sources. By using the potential well method, we give some threshold results on existence and nonexistence of global weak solutions when initial data with energy less than the potential well depth $d$. In the former case, we also show the exponential decay properties of energy functional. We finally obtain some sufficient conditions for the global existence and non-global existence results with high energy initial data. The results of this paper extend some recent results of Han (2018) [11] and Zhou (2019) [32] to the case of PDEs with variable exponent sources.



*Keywords:* Fourth-order parabolic equation; Potential well method; Non-global existence; Global existence; Variable exponents.

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## 1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper, we deal with a solution $u = u(x,t)$ for the following initial-boundary value problem:

$$\begin{cases} u_t + \Delta^2 u - \Delta_{p(x)} u = |u|^{q(x)-2}\, u, & (x,t) \in Q_T, \\ u(x,t) = \dfrac{\partial u}{\partial \nu}(x,t) = 0, & (x,t) \in \Gamma_T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $Q_T := \Omega \times (0,T)$, $\Gamma_T := \partial\Omega \times (0,T)$, $\nu$ is the unit outward normal on $\partial\Omega$ and initial data $u_0 \in H_0^2(\Omega)$. It will also be assumed throughout this paper that $p(x)$ and $q(x)$ are measurable functions on $\Omega$ and satisfy:

$$1 < p^- := \operatorname*{ess\,inf}_{x\in\Omega}\, p(x) \le p^+ := \operatorname*{ess\,sup}_{x\in\Omega}\, p(x) < \begin{cases} \infty, & \text{if} \quad N \le 2, \\ \dfrac{2N}{N-2}, & \text{if} \quad N > 2, \end{cases} \tag{1.2}$$

and

$$\max\left\{2, p^+\right\} < q^- := \operatorname*{ess\,inf}_{x\in\Omega}\, q(x) \le q^+ := \operatorname*{ess\,sup}_{x\in\Omega}\, q(x) < \begin{cases} \infty, & \text{if} \quad N \le 4, \\ \dfrac{2N}{N-4}, & \text{if} \quad N > 4. \end{cases} \tag{1.3}$$

It is well known that the following fourth-order parabolic equations

$$u_t + \Delta^2 u - \nabla \cdot f\left(\nabla u\right) = h\left(x,t,u\right) \tag{1.4}$$

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*Email addresses:* quachuong@gmail.com (Quach Van Chuong), nhanlc@hcmute.edu.vn ( Le Cong Nhan), lxuantruong@ueh.edu.vn (and Le Xuan Truong)

can be used to model a variety of many important physical processes. For example it can be used to described the evolution of the epitaxial growth of nanoscale thin films, see [20, 26, 31] and references therein. The equation (1.4) is also known as classical Cahn–Hilliard equation, an important continuous model for a phase transition with a conservative order parameter, arises from a continuum model for a phase transition in binary systems such as alloys, glasses, and polymer-mixtures; see for example [3, 25, 30].

In mathematical point of view, when the nonlinearities $f$ and $h$ satisfy some constant growth conditions, there have been many results about the existence, uniqueness, and some other properties of the solutions of (1.4), the readers may refer to the bibliography given in [4, 5, 18, 13, 16, 17]. As blow-up property is concerned, in [11] Han used the potential well method proposed by Sattinger [24] (see also [21, 15]) to study the problem

$$\begin{cases} u_t + \Delta^2 u - \Delta_p u = |u|^{q-2} u, & (x,t) \in Q_T, \\ u(x,t) = \dfrac{\partial u}{\partial \nu}(x,t) = 0, & (x,t) \in \Gamma_T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

In that paper, the author showed a threshold result for the solutions to exist globally or to blow up in finite time when the initial energy is subcritical, critical and supercritical initial energy. In addition, the author also studied the decay rate of the $L^2$-norm of global solutions. Then, Zhou [32] studied the exponential decay properties of the energy functional when initial energy $J(u_0) < d_0$, where $d_0$ is a constant given in [11, Lemma 2.1] and is less than the potential well depth $d$.

To our best knowledge, there are few parabolic problems with variable exponent sources. For example, in [22] the author used the eigenfunction argument of Kaplan [12] to study the blow-up property of solutions of the homogeneous Dirichlet problem for the semilinear parabolic equation of the form

$$u_t - \Delta u = f(x,u),$$

where the source term is either

$$f(x,u) = a(x) u^{p(x)} \quad \text{or} \quad f(x,u) = a(x) \int_\Omega u^{q(x)}(y,t)\,dy.$$

Then in [33, 28] the authors established blow-up result for a certain solution of an evolution $m$-Laplace equation involving variable source and suitable positive initial energy

$$u_t - \Delta_m u = |u|^{p(x)-1} u.$$

Recently, in [23] by using the concavity method, the authors established a blow up in finite time result for non-positive initial energy $J(u_0)$ of a fourth-order parabolic equation

$$u_t + \Delta^2 u = u^{q(x)}.$$

Motivated by these papers, we consider in this paper a more general problem with variable exponent nonlinearities of the form

$$\begin{cases} u_t + \Delta^2 u - \Delta_{p(x)} u = |u|^{q(x)-2} u, & (x,t) \in Q_T, \\ u(x,t) = \dfrac{\partial u}{\partial \nu}(x,t) = 0, & (x,t) \in \Gamma_T, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Our results are two folds: Firstly different from the previous results [22, 33, 28, 23] in which the authors only concern about the blow-up property (global nonexistence), our goal in this paper is to establish some sharp results on the existence and nonexistence of global weak solutions for arbitrary initial energy $J(u_0)$. As far as we know, such results in the case of PDEs with variable exponent sources are new. Secondly, the decay rate of $H_0^2$-norm of global solutions which start from potential wells is also concerned. It is noticed that even in the case of constant exponent, Han [11] only proved decay rate of $L^2$-norm of global solutions and Zhou [32] showed the decay of energy functional for $J(u_0) < d_0$. This is not a trivial generalization of similar problems in the constant exponent sources. The substantial difficulties for treating the above problem are caused by the complicated nonlinearities (it is non-homogeneous) and the lack of a maximum principle and comparison principle for fourth-order equations. The key point is to treat the gap between the norm and the integral in variable exponent spaces. Our method presented here can be used to treat the problem in [22, 33, 28, 23].

This paper is organized as follows: In Section 2 we recall some facts about $H_0^2(\Omega)$ space and Ozlicz–Sobolev type spaces; Section 3 study the stationary state of (1.1) and construct the stable sets and unstable sets; Section 4 study the evolution problem and present our main results which its proof are given in the rest of the paper.

## 2. Preliminaries

Let $\Omega$ be as in Section 1. We denote by $\|\cdot\|_r$ the usual norm of the space $L^r(\Omega)$ for $1 \leq r \leq \infty$ and $\langle\cdot,\cdot\rangle$ the usual inner product of the Hilbert space $L^2(\Omega)$. We also denote by $\|\cdot\|_{H_0^2}$ the norm of $H_0^2(\Omega)$. That is

$$\|u\|_{H_0^2} = \sqrt{\|u\|_2^2 + \|\nabla u\|_2^2 + \|\Delta u\|_2^2}.$$

As in [11], $H_0^2(\Omega)$ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle_{H_0^2}$ denoted by

$$\langle u, v\rangle_{H_0^2} = \langle \Delta u, \Delta v\rangle,$$

then it is uniformly convex and the norm $\|\cdot\|_{H_0^2}$ is equivalent to the norm $\|\Delta(\cdot)\|_2$ due to Poincare's inequality.

We next introduce some preliminary results on Lebesgue and Sobolev spaces with variable exponents (see [6, 8, 7, 9, 14]). Denote by $\mathcal{P}(\Omega)$ the set of all measurable functions $p : \Omega \to [1, \infty]$. Define the Lebesgue space with a variable exponent $p(\cdot)$ which is the so-called Nakano space and a special case of Musielak-Orlicz spaces (see [19]), as follows:

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ is measurable}, \ \ \rho(u) := \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty \right\},$$

where $p \in \mathcal{P}(\Omega)$.
The space $L^{p(\cdot)}(\Omega)$ is equipped with the Luxemburg-type norm

$$\|u\|_{p(\cdot)} := \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1 \right\}$$

The following proposition shows the relation between the norm $\|u\|_{p(\cdot)}$ and the modular $\rho(u)$.

**Proposition 1** ([6]). Let $p \in \mathcal{P}(\Omega)$. It holds that

$$\min\left\{ \|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+} \right\} \leq \rho(u) \leq \max\left\{ \|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+} \right\}, \quad \text{for all} \quad u \in L^{p(\cdot)}(\Omega).$$

For $p^+ < \infty$, the dual space of $L^{p(\cdot)}(\Omega)$ is identified with $L^{p'(\cdot)}(\Omega)$ with the dual variable exponent $p' \in \mathcal{P}(\Omega)$ given by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for a.e. } x \in \Omega,$$

where we write $1/\infty = 0$.

The Hölder inequality also holds for variable Lebesgue spaces.

**Proposition 2** (Hölder inequality,[6]). Let $p, q, s \in \mathcal{P}(\Omega)$, it holds that

$$\|uv\|_{s(\cdot)} \leq 2 \|u\|_{p(\cdot)} \|u\|_{q(\cdot)} \quad \text{for all } u \in L^{p(\cdot)}(\Omega), v \in L^{q(\cdot)}(\Omega),$$

provided that

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)} \quad \text{for a.e. } x \in \Omega.$$

**Proposition 3** ([6]). Let $p, q \in \mathcal{P}(\Omega)$. If $p(x) \leq q(x)$ for a.e. $x \in \Omega$, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous.

We next define variable exponent Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ as follows:

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \left( \|u\|_{p(\cdot)}^2 + \|\nabla u\|_{p(\cdot)}^2 \right)^{1/2}.$$

Furthermore, let $W_0^{1,p(\cdot)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. It is noticed that if $1 < p^- \leq p^+ < \infty$, then $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are uniformly convex Banach spaces and therefore they are reflexive.

## 3. Stationary problem and Potential wells

In this section, we consider the stationary solutions of (1.1) which solve the problem

$$\begin{cases} \Delta^2 u - \Delta_{p(x)} u = |u|^{q(x)-2} u & \text{in } \Omega, \\ u(x) = \dfrac{\partial u}{\partial \nu}(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $p(x)$ and $q(x)$ hold (1.2)–(1.3). Consider the energy functional $J$ and the Nehari functional $I$ given by

$$\begin{aligned} J(u) &= \frac{1}{2} \|\Delta u\|_2^2 + \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \,\mathrm{d}x - \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \,\mathrm{d}x, \\ I(u) &= \|\Delta u\|_2^2 + \int_\Omega |\nabla u|^{p(x)} \,\mathrm{d}x - \int_\Omega |u|^{q(x)} \,\mathrm{d}x. \end{aligned}$$

Then $J$ and $I$ are of class $C^1$ over $H_0^2(\Omega)$ and critical points of $J$ are weak solutions of (3.1). Moreover, we can estimate $J$ and $I$ as follows:

$$\begin{aligned} J(u) &\geq \frac{1}{2} \|\Delta u\|_2^2 + \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)} \,\mathrm{d}x - \frac{1}{q^-} \int_\Omega |u|^{q(x)} \,\mathrm{d}x \\ &= \left( \frac{1}{2} - \frac{1}{q^-} \right) \|\Delta u\|_2^2 + \left( \frac{1}{p^+} - \frac{1}{q^-} \right) \int_\Omega |\nabla u|^{p(x)} \,\mathrm{d}x + \frac{1}{q^-} I(u), \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
J(u) &\leq \frac{1}{2}\|\Delta u\|_2^2+\frac{1}{p^-}\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x-\frac{1}{q^+}\int_\Omega |u|^{q(x)}\,\mathrm{d}x \\
&=\left(\frac{1}{2}-\frac{1}{q^+}\right)\|\Delta u\|_2^2+\left(\frac{1}{p^-}-\frac{1}{q^+}\right)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x+\frac{1}{q^+}I(u),
\end{aligned} \tag{3.3}
$$

and

$$
\begin{aligned}
J(u) = &\left(\frac{1}{2}-\frac{1}{q^-}\right)\|\Delta u\|_2^2+\int_\Omega \left(\frac{1}{p(x)}-\frac{1}{q^-}\right)|\nabla u|^{p(x)}\,\mathrm{d}x \\
&+\int_\Omega \left(\frac{1}{q^-}-\frac{1}{q(x)}\right)|u|^{q(x)}\,\mathrm{d}x+\frac{1}{q^-}I(u).
\end{aligned} \tag{3.4}
$$

Let $u \in H_0^2(\Omega)\backslash\{0\}$ and consider the fibering map $\lambda \mapsto j(\lambda) := J(\lambda u)$ for $\lambda > 0$ given by

$$
j(\lambda) = \frac{\lambda^2}{2}\|\Delta u\|_2^2+\int_\Omega \frac{\lambda^{p(x)}}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x-\int_\Omega \frac{\lambda^{q(x)}}{q(x)}|u|^{q(x)}\,\mathrm{d}x.
$$

Then we have the following lemma.

**Lemma 3.1.** Let (1.2)–(1.3) hold and $u \in H_0^2(\Omega)\backslash\{0\}$. Then the following results hold:

(i) $\lim\limits_{\lambda\to 0^+} j(\lambda) = 0$ and $\lim\limits_{\lambda\to\infty} j(\lambda) = -\infty$.

(ii) There exists a $\lambda_* = \lambda_*(u) > 0$ such that $j(\lambda)$ attains the maximum at $\lambda = \lambda_*$. In addition, we have $0 < \lambda_* < 1, \lambda_* = 1$ and $\lambda_* > 1$ provided that $I(u) < 0, I(u) = 0$ and $I(u) > 0$, respectively.

*Proof.* It is easily seen that

$$
j(\lambda) \geq \frac{1}{2}\lambda^2\|\Delta u\|_2^2+\min\left\{\lambda^{p^-},\lambda^{p^+}\right\}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x-\max\left\{\lambda^{q^-},\lambda^{q^+}\right\}\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x,
$$

and

$$
j(\lambda) \leq \frac{1}{2}\lambda^2\|\Delta u\|_2^2+\max\left\{\lambda^{p^-},\lambda^{p^+}\right\}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x-\min\left\{\lambda^{q^-},\lambda^{q^+}\right\}\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x.
$$

This, together with $q^- > \max\{2,p^+\}$ and $\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x > 0$, implies (i). Furthermore, we also have $j(\lambda) > 0$ for sufficiently small $\lambda > 0$. Hence, there exists a $\lambda_* > 0$ such that $j(\lambda_*) = \sup\limits_{\lambda>0} j(\lambda)$. Then by Fermat's Theorem, we obtain $j'(\lambda_*) = 0$. This gives $I(\lambda_* u) = 0$ by the relation $I(\lambda u) = \lambda j'(\lambda)$.

Finally, we prove the last statement of (ii). By the definition of $I$, we obtain

$$
\begin{aligned}
0 &= I(\lambda_* u) \\
&= \lambda_*^2\|\Delta u\|_2^2+\int_\Omega \lambda_*^{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x-\int_\Omega \lambda_*^{q(x)}|u|^{q(x)}\,\mathrm{d}x \\
&= \left(\lambda_*^2-\lambda_*^{q^-}\right)\|\Delta u\|_2^2+\int_\Omega \left(\lambda_*^{p(x)}-\lambda_*^{q^-}\right)|\nabla u|^{p(x)}\,\mathrm{d}x+\int_\Omega \left(\lambda_*^{q^-}-\lambda_*^{q(x)}\right)|u|^{q(x)}\,\mathrm{d}x+\lambda_*^{q^-}I(u),
\end{aligned}
$$

which can be rewritten in the form

$$
\lambda_*^{q^-}I(u) = \left(\lambda_*^{q^-}-\lambda_*^2\right)\|\Delta u\|_2^2+\int_\Omega \left(\lambda_*^{q^-}-\lambda_*^{p(x)}\right)|\nabla u|^{p(x)}\,\mathrm{d}x+\int_\Omega \left(\lambda_*^{q(x)}-\lambda_*^{q^-}\right)|u|^{q(x)}\,\mathrm{d}x.
$$

Since $q^- > \max\{2,p^+\}$, the above equality shows that $0 < \lambda_* < 1, \lambda_* = 1$ and $\lambda_* > 1$ provided that $I(u) < 0, I(u) = 0$ and $I(u) > 0$, respectively. This completes the proof. $\square$

We now define the so-called Nehari manifold associated to the energy functional $J$ by

$$\mathcal{N} = \left\{ u \in H_0^2(\Omega) \setminus \{0\} : I(u) = 0 \right\}.$$

It follows from Lemma 3.1 that $\mathcal{N}$ is not empty set. Thus, we can define

$$d = \inf_{u \in \mathcal{N}} J(u). \tag{3.5}$$

We next give the following lemma, which will play an important role in the proofs of our main results for the low initial energy case.

**Lemma 3.2.** Let (1.2)–(1.3) hold and $u \in H_0^2(\Omega)\backslash\{0\}$. Then we have

$$J(u) - \frac{1}{q^-} I(u) \geq \frac{d}{\max\left\{\lambda_*^2, \lambda_*^{p-}, \lambda_*^{q^+}\right\}},$$

where $\lambda_*$ is as in Lemma 3.1.

*Proof.* For any $u \in H_0^2(\Omega)\backslash\{0\}$. Then by virtue of Lemma 3.1, there exists $\lambda_* \in (0, \infty)$ such that $I(\lambda_* u) = 0$. By the definition of $d$ and replacing $u$ by $\lambda_* u$ in (3.4), one has

$$\begin{aligned}
d &\leq J(\lambda_* u) \\
&= \left(\frac{1}{2} - \frac{1}{q^-}\right) \lambda_*^2 \|\Delta u\|_2^2 + \int_\Omega \lambda_*^{p(x)} \left(\frac{1}{p(x)} - \frac{1}{q^-}\right) |\nabla u|^{p(x)} \,\mathrm{d}x + \int_\Omega \lambda_*^{q(x)} \left(\frac{1}{q^-} - \frac{1}{q(x)}\right) |u|^{q(x)} \,\mathrm{d}x \\
&\leq \max\left\{\lambda_*^2, \lambda_*^{p-}, \lambda_*^{q^+}\right\} \left[\left(\frac{1}{2} - \frac{1}{q^-}\right) \|\Delta u\|_2^2 + \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{q^-}\right) |\nabla u|^{p(x)} \,\mathrm{d}x \right. \\
&\quad \left. + \int_\Omega \left(\frac{1}{q^-} - \frac{1}{q(x)}\right) |u|^{q(x)} \,\mathrm{d}x\right] \\
&= \max\left\{\lambda_*^2, \lambda_*^{p-}, \lambda_*^{q^+}\right\} \left[J(u) - \frac{1}{q^-} I(u)\right].
\end{aligned}$$

This implies the required result. The proof is complete. ☐

Based on the above two lemmas, we can prove the following lemma, which shows that $d$ is positive and is actually attained at some $u \in \mathcal{N}$.

**Lemma 3.3.** Let (1.2)–(1.3) hold. Then we have

(i) $d = \inf\limits_{u \in H_0^2(\Omega)\backslash\{0\}} \sup\limits_{\lambda > 0} J(\lambda u)$.

(ii) $d$ is a positive number.

(iii) There exists $u^* \in \mathcal{N}$, $u^*(x) \geq 0$ a.e. in $\Omega$ such that $J(u^*) = d$.

*Proof.* For any $u \in H_0^2(\Omega)\backslash\{0\}$. By virtue of Lemma 3.1, we have

$$\sup_{\lambda > 0} J(\lambda u) = J(\lambda_* u). \tag{3.6}$$

By the definition of $\mathcal{N}$, it follows from Lemma 3.1 that $\lambda_* u \in \mathcal{N}$. Hence,

$$J(\lambda_* u) \geq \inf_{u \in \mathcal{N}} J(u) = d. \tag{3.7}$$

Combining (3.6) and (3.7), one has

$$\inf_{u \in H_0^2(\Omega)\backslash\{0\}} \sup_{\lambda > 0} J(\lambda u) \geq d. \tag{3.8}$$

On the other hand, for any $u \in \mathcal{N}$, by virtue of Lemma 3.1, one has $\lambda_* = 1$, that is,

$$\sup_{\lambda>0} J(\lambda u) = J(u).$$

Hence,

$$\inf_{u \in H_0^2(\Omega)\backslash\{0\}} \sup_{\lambda>0} J(\lambda u) \leq \inf_{u\in\mathcal{N}} \sup_{\lambda>0} J(\lambda u) = \inf_{u\in\mathcal{N}} J(u) = d. \tag{3.9}$$

We deduce from (3.8) and (3.9) that (i) holds true.

We next prove (ii). Since $q(x)$ satisfies (1.3), $H_0^2(\Omega)$ can be embedded into $L^{q(\cdot)}(\Omega)$ continuously. Denote by $S_{q(.)}$ the optimal embedding constant, i.e.,

$$S_{q(.)} = \sup_{u \in H_0^2(\Omega)\backslash\{0\}} \frac{\|u\|_{q(.)}}{\|\Delta u\|_2}. \tag{3.10}$$

Let any $u \in H_0^2(\Omega)\backslash\{0\}$ such that $I(u) \leq 0$. Then it follows that

$$\begin{aligned}
\|\Delta u\|_2^2 &\leq \int_\Omega |u|^{q(x)}\,\mathrm{d}x \\
&\leq \max\left\{\|u\|_{q(.)}^{q^-}, \|u\|_{q(.)}^{q^+}\right\} \\
&\leq \max\left\{S_{q(.)}^{q^-}\|\Delta u\|_2^{q^-}, S_{q(.)}^{q^+}\|\Delta u\|_2^{q^+}\right\}.
\end{aligned}$$

Taking this fact into account and notice that $\|\Delta u\|_2 > 0$ and $q^- > 2$, we obtain

$$\|\Delta u\|_2 \geq \delta_1, \tag{3.11}$$

where $\delta_1$ is a positive constant given by

$$\delta_1 = \min\left\{S_{q(.)}^{\frac{q^-}{2-q^-}}, S_{q(.)}^{\frac{q^+}{2-q^+}}\right\}.$$

Fix $u \in \mathcal{N}$, we have $u \in H_0^2(\Omega)\backslash\{0\}$ and $I(u) = 0$. By using (3.2) and (3.11), we get

$$\begin{aligned}
J(u) &\geq \left(\frac{1}{2} - \frac{1}{q^-}\right)\|\Delta u\|_2^2 + \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \frac{1}{q^-}I(u) \\
&\geq \left(\frac{1}{2} - \frac{1}{q^-}\right)\|\Delta u\|_2^2 \\
&\geq \left(\frac{1}{2} - \frac{1}{q^-}\right)\delta_1^2.
\end{aligned} \tag{3.12}$$

Then by the definition of $d$, we obtain

$$d \geq \left(\frac{1}{2} - \frac{1}{q^-}\right)\delta_1^2 > 0.$$

Finally, we prove (iii). By (3.5), there exists $\{u_n\}_{n=1}^\infty \subset \mathcal{N}$ is a minimizing sequence of $J$ such that $\lim_{n\to\infty} J(u_n) = d$. Clearly, $|u_n| \in \mathcal{N}$ and $J(|u_n|) = J(u_n)$. For this reason, we may assume that $u_n(x) \geq 0$ a.e. in $\Omega$ for all $n \in \mathbb{N}^*$.

Since $\lim_{n\to\infty} J(u_n) = d$ and using (3.12), we infer that $\{u_n\}$ is bounded in $H_0^2(\Omega)$. Then, since $H_0^2(\Omega)$ is reflexive, $H_0^2(\Omega) \hookrightarrow W_0^{1,p(\cdot)}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ are compact embeddings (by

(1.2) and (1.3)), there exists a sub-sequence of $\{u_n\}$, still denoted by $\{u_n\}$ and a $u^* \in H_0^2(\Omega)$ such that

$$\begin{aligned}
&u_n \rightharpoonup u^* \quad \text{weakly in} \quad H_0^2(\Omega),\\
&u_n \to u^* \quad \text{strongly in} \quad W_0^{1,p(.)}(\Omega),\\
&u_n \to u^* \quad \text{strongly in} \quad L^{q(.)}(\Omega),\\
&u_n(x) \to u^*(x) \quad \text{a.e. in} \quad \Omega.
\end{aligned}$$

Then we have $u^*(x) \geq 0$ a.e in $\Omega$ and

$$\begin{aligned}
\|\Delta u^*\|_2 &\leq \liminf_{n\to\infty} \|\Delta u_n\|_2,\\
\int_\Omega |\nabla u^*|^{p(x)}\,\mathrm{d}x &= \lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x,\\
\int_\Omega \left(\frac{1}{p(x)} - \frac{1}{q^-}\right) |\nabla u^*|^{p(x)}\,\mathrm{d}x &= \lim_{n\to\infty} \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{q^-}\right) |\nabla u_n|^{p(x)}\,\mathrm{d}x,\\
\int_\Omega |u^*|^{q(x)}\,\mathrm{d}x &= \lim_{n\to\infty} \int_\Omega |u_n|^{q(x)}\,\mathrm{d}x.\\
\int_\Omega \left(\frac{1}{q^-} - \frac{1}{q(x)}\right) |u^*|^{q(x)}\,\mathrm{d}x &= \lim_{n\to\infty} \int_\Omega \left(\frac{1}{q^-} - \frac{1}{q(x)}\right) |u_n|^{q(x)}\,\mathrm{d}x.
\end{aligned}$$

Replacing $u$ by $u_n$ in (3.4) and notice that $u_n \in \mathcal{N}$, one has

$$\begin{aligned}
d &= \liminf_{n\to\infty} J(u_n)\\
&= \liminf_{n\to\infty} \left[\left(\frac{1}{2} - \frac{1}{q^-}\right)\|\Delta u_n\|_2^2 + \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{q^-}\right) |\nabla u_n|^{p(x)}\,\mathrm{d}x + \int_\Omega \left(\frac{1}{q^-} - \frac{1}{q(x)}\right) |u_n|^{q(x)}\,\mathrm{d}x\right]\\
&\geq \left(\frac{1}{2} - \frac{1}{q^-}\right)\|\Delta u^*\|_2^2 + \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{q^-}\right) |\nabla u^*|^{p(x)}\,\mathrm{d}x + \int_\Omega \left(\frac{1}{q^-} - \frac{1}{q(x)}\right) |u^*|^{q(x)}\,\mathrm{d}x\\
&= J(u^*) - \frac{1}{q^-} I(u^*).
\end{aligned} \tag{3.13}$$

Suppose that $I(u^*) < 0$. Then by virtue of Lemma 3.1 and 3.2, there exists $\lambda_* \in (0, 1)$ such that

$$J(u^*) - \frac{1}{q^-} I(u^*) \geq \frac{d}{\max\left\{\lambda_*^2, \lambda_*^{p-}, \lambda_*^{q^+}\right\}} > d.$$

This contradicts (3.13), and so

$$I(u^*) \geq 0. \tag{3.14}$$

Since $u_n \in \mathcal{N}$, we have $I(u_n) = 0$. Then it follows that

$$\begin{aligned}
0 = \liminf_{n\to\infty} I(u_n) &= \liminf_{n\to\infty} \left(\|\Delta u_n\|_2^2 + \int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x - \int_\Omega |u_n|^{q(x)}\,\mathrm{d}x\right)\\
&\geq \|\Delta u^*\|_2^2 + \int_\Omega |\nabla u^*|^{p(x)}\,\mathrm{d}x - \int_\Omega |u^*|^{q(x)}\,\mathrm{d}x\\
&= I(u^*),
\end{aligned}$$

which, together with (3.14), implies that $I(u^*) = 0$. We now prove $u^* \in \mathcal{N}$. It remains to show that $u^* \neq 0$. Indeed, since $u_n \in \mathcal{N}$ and (3.11), we obtain

$$\int_\Omega |u_n|^{q(x)}\,\mathrm{d}x = \|\Delta u_n\|_2^2 + \int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x \geq \delta_1^2.$$

Passing to the limit, we have

$$\int_{\Omega} |u^*|^{q(x)}\,\mathrm{d}x \geq \delta_1^2 > 0,$$

which gives $u^* \neq 0$. Hence, $u^* \in \mathcal{N}$ and therefore $J(u^*) \geq d$. By (3.13) and $I(u^*) = 0$, we have $J(u^*) \leq d$. So, $J(u^*) = d$. The proof is complete. $\square$

We now can define the so-called stable set $\mathcal{W}$ and unstable set $\mathcal{U}$ which is similar to Sattinger [24], Payne and Sattinger [21].

$$\mathcal{W} = \left\{u \in H_0^2(\Omega) : J(u) < d, I(u) > 0\right\} \cup \{0\},$$
$$\mathcal{U} = \left\{u \in H_0^2(\Omega) : J(u) < d, I(u) < 0\right\},$$

We also introduce

$$\mathcal{N}_- = \left\{u \in H_0^2(\Omega) : I(u) < 0\right\}, \quad \mathcal{N}_+ = \left\{u \in H_0^2(\Omega) : I(u) > 0\right\},$$

and the open sublevels of $J$

$$J^k = \left\{u \in H_0^2(\Omega) : J\,(u) < k\right\}.$$

The variational characterization of $d$ also shows that

$$\mathcal{N}_k := \mathcal{N} \cap J^k \neq \emptyset \quad \text{for all} \quad k > d.$$

For $k > d$, we now define

$$\lambda_k = \inf\left\{\|u\|_2 : u \in \mathcal{N}_k\right\}, \quad \text{and} \quad \Lambda_k = \sup\left\{\|u\|_2 : u \in \mathcal{N}_k\right\}. \tag{3.15}$$

It is obviously that

$$k \mapsto \lambda_k \text{ is non-increasing}, \quad \text{and} \quad k \mapsto \Lambda_k \text{ is non-decreasing}.$$

The next lemma shows that $\lambda_k$ and $\Lambda_k$ are finite positive numbers, and therefore the result of Theorem 4.8 is nontrivial.

**Lemma 3.4.** Let (1.2)–(1.3) hold. Then for any $k > d$, $\lambda_k$ and $\Lambda_k$ defined in (3.15) satisfy

$$0 < \lambda_k \leq \Lambda_k < \infty.$$

*Proof.* Firstly, we prove $\Lambda_k < \infty$. For any $k > d$ and $u \in \mathcal{N}_k$, we have $J\,(u) < k$ and $I\,(u) = 0$. Then by (3.2) and using the embedding $H_0^2\,(\Omega) \hookrightarrow L^2\,(\Omega)$, we get

$$\begin{aligned} k > J\,(u) &\geq \left(\frac{1}{2} - \frac{1}{q^-}\right)\|\Delta u\|_2^2 + \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\int_{\Omega}|\nabla u|^{p(x)}\,\mathrm{d}x + \frac{1}{q^-}I\,(u) \\ &\geq \left(\frac{1}{2} - \frac{1}{q^-}\right)\|\Delta u\|_2^2 \\ &\geq \left(\frac{1}{2} - \frac{1}{q^-}\right)S_2^{-2}\,\|u\|_2^2, \end{aligned} \tag{3.16}$$

which yields

$$\|u\|_2 \leq S_2\sqrt{\frac{2kq^-}{q^- - 2}},$$

where $S_2 > 0$ is the optimal embedding constant, i.e.,

$$S_2 = \sup_{u \in H_0^2(\Omega)\backslash\{0\}} \frac{\|u\|_2}{\|\Delta u\|_2}. \tag{3.17}$$

This shows that

$$\Lambda_k \leq S_2 \sqrt{\frac{2kq^-}{q^- - 2}} < \infty.$$

Secondly, we prove $\lambda_k > 0$. By the Gagliardo–Nirenberg inequality, there exists a positive constant $A_0$ depending only on $\Omega, N, q^-$ and $q^+$ such that

$$\|u\|_{q^-}^{q^-} \leq A_0 \|\Delta u\|_2^{\theta^- q^-} \|u\|_2^{(1-\theta^-)q^-},$$

and

$$\|u\|_{q^+}^{q^+} \leq A_0 \|\Delta u\|_2^{\theta^+ q^+} \|u\|_2^{(1-\theta^+)q^+},$$

where $\theta^\pm = \dfrac{N\left(q^\pm - 2\right)}{4q^\pm} \in (0;1)$ by (1.3).
Then, since $u \in \mathcal{N}_k$ and $\mathcal{N}_k \subset \mathcal{N}$, it follows that

$$\begin{aligned}
\|\Delta u\|_2^2 &\leq \int_\Omega |u|^{q(x)}\,\mathrm{d}x \\
&\leq \int_\Omega \left(|u|^{q^-} + |u|^{q^+}\right)\mathrm{d}x \\
&\leq 2\max\left\{\|u\|_{q^-}^{q^-}, \|u\|_{q^+}^{q^+}\right\} \\
&\leq 2A_0 \max\left\{\|\Delta u\|_2^{\theta^- q^-} \|u\|_2^{(1-\theta^-)q^-}, \|\Delta u\|_2^{\theta^+ q^+} \|u\|_2^{(1-\theta^+)q^+}\right\}.
\end{aligned}$$

Taking this fact into account and notice that $\|\Delta u\|_2 > 0$ and $\theta^\pm < 1$, we obtain

$$\|u\|_2 \geq \min\left\{(2A_0)^{\frac{1}{(\theta^- - 1)q^-}} \|\Delta u\|_2^{\frac{2-\theta^- q^-}{(1-\theta^-)q^-}}, (2A_0)^{\frac{1}{(\theta^+ - 1)q^+}} \|\Delta u\|_2^{\frac{2-\theta^+ q^+}{(1-\theta^+)q^+}}\right\}. \tag{3.18}$$

On the other hand, it follows from (3.11) and (3.16) that

$$\delta_1 \leq \|\Delta u\|_2 \leq \sqrt{\frac{2kq^-}{q^- - 2}} := \delta_2, \quad \text{for all} \quad u \in \mathcal{N}_k.$$

This, together with (3.18), implies

$$\begin{aligned}
\|u\|_2 \geq \min\Bigg\{&(2A_0)^{\frac{1}{(\theta^- - 1)q^-}} \min\left\{\delta_1^{\frac{2-\theta^- q^-}{(1-\theta^-)q^-}}, \delta_2^{\frac{2-\theta^- q^-}{(1-\theta^-)q^-}}\right\}, \\
&(2A_0)^{\frac{1}{(\theta^+ - 1)q^+}} \min\left\{\delta_1^{\frac{2-\theta^+ q^+}{(1-\theta^+)q^+}}, \delta_2^{\frac{2-\theta^+ q^+}{(1-\theta^+)q^+}}\right\}\Bigg\} > 0.
\end{aligned}$$

Hence, $\lambda_k > 0$ by the definition of $\lambda_k$. This completes the proof. $\square$

Finally, we give the following lemma, which is necessary for our proofs of the main results in case of the high initial energy.

**Lemma 3.5.** Let (1.2)–(1.3) hold. Then we have

(i) 0 is away from both $\mathcal{N}$ and $\mathcal{N}_-$, that is, $dist(0,\mathcal{N}) > 0$ and $dist(0,\mathcal{N}_-) > 0$.

(ii) The set $\mathcal{N}_+ \cap J^k$ is bounded in $H_0^2(\Omega)$ for any $k > 0$.

*Proof.* By (3.11), it is easy to see that

$$dist\,(0,\mathcal{N}) = \inf_{u\in\mathcal{N}} \|\Delta u\|_2 \geq \delta_1 > 0,$$

and

$$dist\,(0,\mathcal{N}_-) = \inf_{u\in\mathcal{N}_-} \|\Delta u\|_2 \geq \delta_1 > 0.$$

We now prove (ii). For any $u \in \mathcal{N}_+ \cap J^k$, we have $J(u) < k$ and $I(u) > 0$. Then by using (3.2), it follows that

$$\begin{aligned} k > J(u) &\geq \left(\frac{1}{2} - \frac{1}{q^-}\right)\|\Delta u\|_2^2 + \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \frac{1}{q^-} I\,(u) \\ &\geq \left(\frac{1}{2} - \frac{1}{q^-}\right)\|\Delta u\|_2^2, \end{aligned}$$

which implies that

$$\|\Delta u\|_2 < \sqrt{\frac{2kq^-}{q^- - 2}}.$$

This completes the proof. □

## 4. Evolution problem

We first give the precise meaning of solution to problem (1.1).

**Definition 4.1.** Let $T > 0$, a function $u = u(t) \in L^\infty\left(0,T;H_0^2\,(\Omega)\right)$ with $u_t \in L^2\left(0,T;L^2\,(\Omega)\right)$ is said to be a *weak solution* to (1.1) in $\Omega\times[0,T)$, if $u(0) = u_0 \in H_0^2\,(\Omega)$ and satisfies

$$\langle u_t, v\rangle + \langle \Delta u, \Delta v\rangle + \left\langle |\nabla u|^{p(x)-2}\,\nabla u, \nabla v\right\rangle = \left\langle |u|^{q(x)-2}\,u, v\right\rangle,$$

for all $v \in H_0^2\,(\Omega)$. In addition, $u$ is called a *strong solution* to (1.1) if it is a weak solution and satisfies the energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 = -2I\,(u(t))\,, \tag{4.1}$$

and the conservation law

$$\int_{t_1}^{t_2} \left\|u'(s)\right\|_2^2 \mathrm{d}s + J\,(u(t_2)) = J\,(u(t_1))\,, \quad 0 \leq t_1 \leq t_2 < T. \tag{4.2}$$

**Remark 4.2.**
- Under the assumption $u_0 \in H_0^2\,(\Omega)$, by using the standard Galerkin's method as in [11, 32], we can prove immediately the existence of local weak solutions $u(t)$. In addition, if $u(t)$ is a strong solution to (1.1) then it follows from (4.2) and the equi-integrability of $u_t$ that $J(u(t))$ is continuous functional which implies $u \in C\left([0,T);H_0^2\,(\Omega)\right)$. In the rest of this paper, we shall prove the global existence and nonexistence of strong solution to (1.1).

- Regarding the uniqueness of solution to (1.1), it can be obtained under some suitable assumptions on the initial data $u_0$, $p(\cdot)$ and $q(\cdot)$. For example, in [11] Han proved the uniqueness of bounded weak solution ($L^\infty$-bounded solution), such kind of solution can be obtained by either in one dimensional space or $p(\cdot) > N$.

- It is also noticed that the proof of global nonexistence generally does not imply finite time blow-up of the solution as pointed in [1]. However, if one can couple the global nonexistence with the continuation principle, then the global nonexistence implies finite time blow-up. In this paper, we will assume the existence and uniqueness of the local strong solution and refer the *continuation principle* as follows: If $u(t)$ is a local strong solution to (1.1) on $[0, T)$ such that

$$\sup_{t\in[0,T)} \|u(t)\|_{H_0^2} < \infty$$

then it can be continued as a solution to a larger interval $[0, T')$ with $T' > T$. Furthermore, if we let $T_{\max}$ be the maximal existence time of the solution $u(t)$ then either the life span $T_{\max} = \infty$ or $T_{\max}$ is finite and

$$\lim_{t\to T_{\max}} \|u(t)\|_{H_0^2} = \infty.$$

The next lemma shows the invariant of stable and unstable sets.

**Lemma 4.3.** Let (1.2)–(1.3) hold and $J(u_0) < d$. Then we possess the following statements:

(i) If $I(u_0) < 0$, then $I(u(t)) < 0$ for all $t \in [0, T_{\max})$.

(ii) If $I(u_0) \geq 0$, then $I(u(t)) \geq 0$ for all $t \in [0, T_{\max})$.

*Proof.* It is first noticed that $u(t) \notin \mathcal{N}$, for all $t \in [0, T_{\max})$ since $J(u(t)) \leq J(u_0)) < d$.

For (i). Assume the contrary that there exists $t_0 \in (0, T_{\max})$ such that $I(u(t)) < 0$ for all $t \in [0, t_0)$ and $I(u(t_0)) = 0$. Then by (3.11), we get $\|\Delta u(t)\|_2 \geq \delta_1$, for all $t \in [0, t_0)$. Letting $t \to t_0$, we have $\|\Delta u(t_0)\|_2 \geq \delta_1$, which gives $u(t_0) \neq 0$. Hence, $u(t_0) \in \mathcal{N}$. We thus arrive at a contradiction.

For (ii). By contradiction, we assume that there exists $t_1 \in (0, T_{\max})$ such that $I(u(t_1)) < 0$. This and $I(u_0) \geq 0$ imply that there exists $t_2 \in [0, t_1)$ such that $I(u(t_2)) = 0$. It gives $u(t_2) = 0$ due to $u(t_2) \notin \mathcal{N}$. Then we get $u(t) = 0$ for $t \in [t_2, T_{\max})$. Thus $u(t_1) = 0$. This contradicts $I(u(t_1)) < 0$. The proof is complete. □

We introduce the sets

$$\mathcal{S} = \left\{\phi \in H_0^2(\Omega) : \phi \text{ is a stationary solution of (1.1)}\right\},$$

and define the $\omega$-limit set $\omega(u_0)$ of the initial data $u_0 \in W_0^{1,p(\cdot)}(\Omega)$ by

$$\omega(u_0) = \left\{w \in H_0^2(\Omega) : \exists\{t_n\} \text{ with } t_n \to \infty \text{ such that } u(t_n) \to w\right\}.$$

Let $u(t)$ be a solution to (1.1) associated with $u_0 \in H_0^2(\Omega)$ on the maximal existence time interval $[0, T_{\max})$. We then introduce the sets

$$\mathcal{G} = \left\{u_0 \in H_0^2(\Omega) : u(t) \text{ exists globally, i.e., } T_{\max} = \infty\right\},$$
$$\mathcal{G}_0 = \left\{u_0 \in \mathcal{G} : u(t) \to 0 \text{ in } H_0^2(\Omega) \text{ as } t \to \infty\right\},$$
$$\mathcal{B} = \left\{u_0 \in H_0^2(\Omega) : u(t) \text{ blows up in finite time, i.e., } T_{\max} < \infty\right\}.$$

Our main results read as follows.

**Theorem 4.4.** Let (1.2)–(1.3) hold. If $J(u_0) < d$ and $I(u_0) \geq 0$, then the maximal existence time $T_{\max} = \infty$. Moreover, $u(t)$ holds the following decay estimates:

$$\|u(t)\|_2 \leq \|u_0\|_2 \, e^{-\alpha t},$$

$$\|\Delta u(t)\|_2 \leq \sqrt{\frac{2q^-}{q^- - 2}\left(J(u_0) + \|u_0\|_2^2\right)} e^{-\beta t},$$

$$\sqrt{J(u(t)) + \|u(t)\|_2^2} \leq \sqrt{J(u_0) + \|u_0\|_2^2} e^{-\beta t},$$

where $\alpha$ and $\beta$ are some positive constants.

**Theorem 4.5.** Let (1.2)–(1.3) hold. Then we possess

(i) If $u_0 \in H_0^2(\Omega) \backslash \{0\}$ holds $J(u_0) \leq 0$ then $T_{\max} < \infty$. Furthermore, we can get an upper bound for the maximal existence time

$$T_{\max} \leq C \max\left\{\|u_0\|_2^{2-q^-}, \|u_0\|_2^{2-q^+}\right\},$$

where

$$C = \frac{q^- \max\left\{S_{q(.),2}^{q^-}, S_{q(.),2}^{q^+}\right\}}{(q^- - 2)\left(q^- - \max\{2, p^+\}\right)} > 0, \tag{4.3}$$

and $S_{q(.),2}$ is the optimal embedding constant of $L^{q(\cdot)}(\Omega) \hookrightarrow L^2(\Omega)$ when $q^- > 2$, i.e.,

$$S_{q(.),2} = \sup_{u \in L^{q(\cdot)}(\Omega)\backslash\{0\}} \frac{\|u\|_2}{\|u\|_{q(.)}}. \tag{4.4}$$

(ii) If $0 < J(u_0) < d$ and $I(u_0) < 0$, then the maximal existence time $T_{\max} < \infty$.

**Remark 4.6.** As a consequence of Theorem 4.4 and 4.5 we have a sharp result in case of $J(u_0) < d$, that is, the strong solution to (1.1) exists globally and blows up in finite time provided that $I(u_0) \geq 0$ and $I(u_0) < 0$, respectively.

Theorem 4.4 shows that any global solution which starts from the potential wells $\mathcal{W}$ tends to zero. And the next theorem shows the asymptotic behavior of any global solution of (1.1).

**Theorem 4.7.** Let (1.2)–(1.3) hold and $u(t)$ be a global solution of (1.1). Then there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ and $\phi \in \mathcal{S}$ such that

$$\lim_{n\to\infty} \|\Delta u(t_n) - \Delta\phi\|_2 = 0.$$

Our next result gives an abstract criterion for vanishing and global nonexistence of solutions to (1.1) in terms of the variational values $\lambda_k$ and $\Lambda_k$.

**Theorem 4.8.** Let (1.2)–(1.3) hold and $J(u_0) > d$. If $u_0 \in \mathcal{N}_+$ and $\|u_0\|_2 \leq \lambda_{J(u_0)}$, then $u_0 \in \mathcal{G}_0$. If $u_0 \in \mathcal{N}_-$ and $\|u_0\|_2 \geq \Lambda_{J(u_0)}$, then $u_0 \in \mathcal{B}$.

As a consequence, one has a characterization on the data $u_0$ with arbitrary high energy $J(u_0)$ that leads to blow-up in finite time phenomena.

**Theorem 4.9.** Let (1.2)–(1.3) hold and assume that $u_0 \in H_0^2(\Omega)$ holds $J(u_0) > d$ and

$$\|u_0\|_2^2 \geq \frac{2q^- S_2^2}{q^- - 2} J(u_0), \tag{4.5}$$

then $u_0 \in \mathcal{N}_- \cap \mathcal{B}$. Here $S_2$ is the constant given in (3.17).

Finally we can exhibit a class of initial data in $\mathcal{N}_-$ with arbitrarily high energy which gives rise to blow-up.

**Theorem 4.10.** Let (1.2)–(1.3) hold. Then for any $M > 0$, there exists $u_M \in \mathcal{N}_-$ such that $J(u_M) \geq M$ and $u_M \in \mathcal{B}$.

## 5. Proof of Theorem 4.4

Let $u(t) := u(x,t)$ be a solution of (1.1) on the interval $[0, T_{\max})$ associated with to the initial data $u_0$. We first prove the uniform boundedness in time of $u(t)$ in $H_0^2(\Omega)$, which implies $T_{\max} = \infty$ by the continuation principle. Indeed, since $J(u_0) < d$ and $I(u_0) \geq 0$, by virtue of Lemma 4.3 we have that $I(u(t)) \geq 0$. Then by using the non-increasing property of $J(u(t))$ and (3.2), we get

$$
\begin{aligned}
J(u_0) \geq J(u(t)) &\geq \left(\frac{1}{2} - \frac{1}{q^-}\right) \|\Delta u(t)\|_2^2 + \left(\frac{1}{p^+} - \frac{1}{q^-}\right) \int_\Omega |\nabla u(t)|^{p(x)} \, \mathrm{d}x + \frac{1}{q^-} I(u(t)) \\
&\geq \left(\frac{1}{2} - \frac{1}{q^-}\right) \|\Delta u(t)\|_2^2, \qquad (5.6)
\end{aligned}
$$

which implies

$$
\|\Delta u(t)\|_2 \leq \sqrt{\frac{2q^- J(u_0)}{q^- - 2}} := C_*.
$$

We next show the decay estimates of $u(t)$. If $u(t_0) = 0$ for some $t_0 \geq 0$, then we have $u(t) = 0$ for all $t \geq t_0$, and the proof is complete. So we may assume $u(t) \neq 0$ for all $t \geq 0$. Then due to $I(u(t)) \geq 0$, by virtue of Lemma 3.1, there exists $\lambda_* \geq 1$ such that $I(\lambda_* u(t)) = 0$. And therefore

$$
\begin{aligned}
\lambda_*^{q^-} I(u(t)) &= \lambda_*^{q^-} I(u(t)) - I(\lambda_* u(t)) \\
&= \left(\lambda_*^{q^-} - \lambda_*^2\right) \|\Delta u(t)\|_2^2 + \int_\Omega \left(\lambda_*^{q^-} - \lambda_*^{p(x)}\right) |\nabla u(t)|^{p(x)} \, \mathrm{d}x \\
&\quad + \int_\Omega \left(\lambda_*^{q(x)} - \lambda_*^{q^-}\right) |u(t)|^{q(x)} \, \mathrm{d}x \\
&\geq \left(\lambda_*^{q^-} - \lambda_*^2\right) \|\Delta u(t)\|_2^2 + \left(\lambda_*^{q^-} - \lambda_*^{p^+}\right) \int_\Omega |\nabla u(t)|^{p(x)} \, \mathrm{d}x.
\end{aligned}
$$

Dividing the above inequality by $\lambda_*^{q^-}$, we get

$$
I(u(t)) \geq \left(1 - \lambda_*^{2-q^-}\right) \|\Delta u(t)\|_2^2 + \left(1 - \lambda_*^{p^+ - q^-}\right) \int_\Omega |\nabla u(t)|^{p(x)} \, \mathrm{d}x. \qquad (5.7)
$$

We next estimate for $\lambda_*$. By applying Lemma 3.2 and notice that $\lambda_* \geq 1$, one has

$$
J(u(t)) - \frac{1}{q^-} I(u(t)) \geq \frac{d}{\max\left\{\lambda_*^2, \lambda_*^{p-}, \lambda_*^{q^+}\right\}} = \frac{d}{\lambda_*^{q^+}}. \qquad (5.8)
$$

On the other hand, by using the non-increasing property of $J(u(t))$ and notice that $I(u(t)) \geq 0$, we have

$$
J(u(t)) - \frac{1}{q^-} I(u(t)) \leq J(u_0).
$$

This together with (5.8), implies that

$$
\lambda_* \geq \left(\frac{d}{J(u_0)}\right)^{\frac{1}{q^+}} > 1. \qquad (5.9)
$$

It follows from (5.7) and (5.9) that

$$
I(u(t)) \geq \left(1 - \left(\frac{d}{J(u_0)}\right)^{\frac{2-q^-}{q^+}}\right) \|\Delta u(t)\|_2^2 + \left(1 - \left(\frac{d}{J(u_0)}\right)^{\frac{p^+ - q^-}{q^+}}\right) \int_\Omega |\nabla u(t)|^{p(x)} \, \mathrm{d}x,
$$

which yields

$$I(u(t)) \geq C_1 \|\Delta u(t)\|_2^2 \quad \text{and} \quad I(u(t)) \geq C_2 \int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x, \tag{5.10}$$

where $C_1$ and $C_2$ are positive constants given by

$$C_1 = 1 - \left(\frac{d}{J(u_0)}\right)^{\frac{2-q^-}{q^+}} \quad \text{and} \quad C_2 = 1 - \left(\frac{d}{J(u_0)}\right)^{\frac{p^+-q^-}{q^+}}.$$

We now consider the exponential decay of $\|u(t)\|_2$. Since (5.10), it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 &= -2I(u(t))\\ &\leq -2C_1\|\Delta u(t)\|_2^2\\ &\leq -2C_1 S_2^{-2}\|u(t)\|_2^2,\end{aligned}$$

where $S_2$ is the constant given in (3.17). This implies that

$$\|u(t)\|_2 \leq \|u_0\|_2 e^{-\alpha t},$$

where $\alpha = C_1 S_2^{-2} > 0$.

We next consider the exponential decay of $J(u(t))$ and $\|\Delta u(t)\|_2$. Using (3.3), we get

$$J(u(t)) \leq \left(\frac{1}{2} - \frac{1}{q^+}\right)\|\Delta u(t)\|_2^2 + \left(\frac{1}{p^-} - \frac{1}{q^+}\right)\int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x + \frac{1}{q^+}I(u(t)).$$

This together with (5.10) immediately yields

$$J(u(t)) \leq C_3 I(u(t)), \tag{5.11}$$

where

$$C_3 = \frac{1}{C_1}\left(\frac{1}{2} - \frac{1}{q^+}\right) + \frac{1}{C_2}\left(\frac{1}{p^-} - \frac{1}{q^+}\right) + \frac{1}{q^+} > 0.$$

Let us define an auxiliary function $L(t)$ by

$$L(t) = J(u(t)) + \|u(t)\|_2^2, \quad \text{for } t \geq 0. \tag{5.12}$$

Then by (5.6) and (5.12), we get

$$L(t) \leq J(u(t)) + S_2^2\|\Delta u(t)\|_2^2 \leq C_4 J(u(t)), \tag{5.13}$$

here $C_4 = 1 + \dfrac{2q^-}{q^- - 2}S_2^2 > 0$ and $S_2$ is the constant given in (3.17).
It follows from (5.11), (5.12) and (5.13) that

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) = -\left\|u'(t)\right\|_2^2 - 2I(u(t)) \leq -\frac{2}{C_3}J(u(t)) \leq -\frac{2}{C_3 C_4}L(t),$$

which implies that

$$L(t) \leq L(0)e^{-2\beta t},$$

where $\beta = \dfrac{1}{C_3C_4} > 0$. The above inequality can be rewritten as

$$J(u(t)) + \|u(t)\|_2^2 \le \left(J(u_0) + \|u_0\|_2^2\right) e^{-2\beta t}. \tag{5.14}$$

By (5.6) and (5.14), we get

$$\begin{aligned}\|\Delta u(t)\|_2^2 &\le \frac{2q^-}{q^- - 2} J(u(t)) \\ &\le \frac{2q^-}{q^- - 2}\left(J(u_0) + \|u_0\|_2^2\right) e^{-2\beta t}.\end{aligned}$$

The proof is complete.

## 6. Proof of Theorem 4.5

We consider two following case by using different methods:

**Case 1:** $u_0 \in H_0^2(\Omega)\backslash\{0\}$ **with** $J(u_0) \le 0$**.** We define the function $f(t)$ by

$$f(t) = \|u(t)\|_2^2, \quad \text{for all } t \in [0, T_{\max}).$$

By the definition of $J$ and $I$, we have

$$\begin{aligned}J(u(t)) &\ge \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{p^+}\int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x - \frac{1}{q^-}\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x \\ &\ge \frac{1}{\max\{2,p^+\}}\left(\|\Delta u(t)\|_2^2 + \int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x\right) - \frac{1}{q^-}\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x \\ &= \left(\frac{1}{\max\{2,p^+\}} - \frac{1}{q^-}\right)\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x + \frac{1}{\max\{2,p^+\}} I(u(t)).\end{aligned}$$

Taking this fact into account and notice that $J(u(t)) \le J(u_0) \le 0$, one has

$$\begin{aligned}f'(t) &= -2I(u(t)) \\ &\ge -2\max\{2,p^+\}J(u(t)) + 2\left(1 - \frac{\max\{2,p^+\}}{q^-}\right)\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x \\ &\ge 2\left(1 - \frac{\max\{2,p^+\}}{q^-}\right)\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x.\end{aligned} \tag{6.1}$$

From this and $q^- > \max\{2,p^+\}$, we get

$$f'(t) \ge 0, \quad \text{for all } t \in [0, T_{\max}),$$

which implies that

$$f(t) \ge f(0) = \|u_0\|_2^2 > 0, \quad \text{for all } t \in [0, T_{\max}). \tag{6.2}$$

Then by (6.2), we can estimate $\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x$ as follows:

$$\begin{aligned}\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x &\ge \min\left\{\|u(t)\|_{q(.)}^{q^-}, \|u(t)\|_{q(.)}^{q^+}\right\} \\ &\ge \min\left\{S_{q(.),2}^{-q^-}\|u(t)\|_2^{q^-}, S_{q(.),2}^{-q^+}\|u(t)\|_2^{q^+}\right\} \\ &\ge \min\left\{S_{q(.),2}^{-q^-}, S_{q(.),2}^{-q^+}\right\}\min\left\{\|u(t)\|_2^{q^-}, \|u(t)\|_2^{q^+}\right\} \\ &= \min\left\{S_{q(.),2}^{-q^-}, S_{q(.),2}^{-q^+}\right\}\min\left\{1, f^{\frac{q^+-q^-}{2}}(t)\right\} f^{\frac{q^-}{2}}(t) \\ &\ge \min\left\{S_{q(.),2}^{-q^-}, S_{q(.),2}^{-q^+}\right\}\min\left\{1, \|u_0\|_2^{q^+-q^-}\right\} f^{\frac{q^-}{2}}(t),\end{aligned} \tag{6.3}$$

where $S_{q(.),2}$ is defined in (4.4).
It follows from (6.1) and (6.3) that

$$f'(t) \geq C_0 f^{\frac{q^-}{2}}(t), \tag{6.4}$$

where

$$C_0 = 2\left(1 - \frac{\max\{2, p^+\}}{q^-}\right) \min\left\{S_{q(.),2}^{-q^-}, S_{q(.),2}^{-q^+}\right\} \min\left\{1, \|u_0\|_2^{q^+-q^-}\right\} > 0.$$

Since $f(t) > 0$, dividing the inequality (6.4) by $f^{\frac{q^-}{2}}(t)$, we get

$$f'(t) f^{-\frac{q^-}{2}}(t) \geq C_0.$$

Integrating the above inequality over $[0, t]$, one has

$$f^{1-\frac{q^-}{2}}(t) \leq f^{1-\frac{q^-}{2}}(0) - \left(\frac{q^-}{2} - 1\right) C_0 t, \quad \text{for all } t \in [0, T_{\max}).$$

This and $f^{1-\frac{q^-}{2}}(t) > 0$ imply

$$t < \frac{2}{(q^- - 2)C_0} \|u_0\|_2^{2-q^-}, \quad \text{for all } t \in [0, T_{\max}).$$

Thus,

$$\begin{aligned} T_{\max} &\leq \frac{2}{(q^- - 2)C_0} \|u_0\|_2^{2-q^-} \\ &= C \max\left\{\|u_0\|_2^{2-q^-}, \|u_0\|_2^{2-q^+}\right\}, \end{aligned}$$

where $C$ is the constant given in (4.3). The proof is complete.

**Case 2: $0 < J(u_0) < d$ and $I(u_0) < 0$.** By contradiction, we assume that $T_{\max} = \infty$. Thanks to $I(u_0) < 0$, by Lemma 4.3 we have $I(u(t)) < 0$. Then by virtue of Lemma 3.1 and 3.2, there exists $\lambda_* \in (0, 1)$ such that

$$J(u(t)) - \frac{1}{q^-} I(u(t)) \geq \frac{d}{\max\left\{\lambda_*^2, \lambda_*^{p-}, \lambda_*^{q^+}\right\}} > d,$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|_2^2 = -2I(u(t)) > 2q^- (d - J(u(t))) \geq 2q^- (d - J(u_0)).$$

Then we have

$$\begin{aligned} \|u(t)\|_2^2 &= \|u_0\|_2^2 + \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \|u(s)\|_2^2 \,\mathrm{d}s \\ &\geq \|u_0\|_2^2 + 2q^- (d - J(u_0))\, t. \end{aligned} \tag{6.5}$$

From this and $J(u_0) < d$, we obtain $\lim\limits_{t\to\infty} \|u(t)\|_2^2 = \infty$. Hence, we can choose sufficiently large $t_0 > 0$ such that

$$\|u(t_0)\|_2^2 > \frac{q^-}{q^- - 2} \|u_0\|_2^2.$$

Let

$$T = \frac{\int_0^{t_0} \|u(s)\|_2^2 \,\mathrm{d}s}{\left(\frac{q^-}{2} - 1\right)\left(\|u(t_0)\|_2^2 - \frac{q^-}{q^- - 2}\|u_0\|_2^2\right)} + t_0 \geq t_0 > 0. \tag{6.6}$$

We now define the auxiliary function $F : [0, T] \to (0, \infty)$ by

$$F(t) = \int_0^t \|u(s)\|_2^2 \,\mathrm{d}s + (T - t)\|u_0\|_2^2. \tag{6.7}$$

Then we have

$$F'(t) = \|u(t)\|_2^2 - \|u_0\|_2^2 = 2\int_0^t \langle u'(s), u(s)\rangle \,\mathrm{d}s,$$

and

$$\begin{aligned} F''(t) = 2\langle u'(t), u(t)\rangle &= -2I(u(t)) \\ &> 2q^-\left(d - J(u(t))\right) \\ &= 2q^-\left(d - J(u_0)\right) + 2q^-\int_0^t \|u'(s)\|_2^2 \,\mathrm{d}s \\ &\geq 2q^-\int_0^t \|u'(s)\|_2^2 \,\mathrm{d}s. \end{aligned} \tag{6.8}$$

We deduce from (6.7) and (6.8) that

$$F(t)F''(t) \geq 2q^-\int_0^t \|u'(s)\|_2^2 \,\mathrm{d}s \int_0^t \|u(s)\|_2^2 \,\mathrm{d}s. \tag{6.9}$$

On the other hand, by Cauchy–Schwarz inequality, we have

$$\int_0^t \|u'(s)\|_2^2 \,\mathrm{d}s \int_0^t \|u(s)\|_2^2 \,\mathrm{d}s \geq \left(\int_0^t \langle u'(s), u(s)\rangle \,\mathrm{d}s\right)^2 = \frac{1}{4}\left(F'(t)\right)^2. \tag{6.10}$$

Combining (6.9)–(6.10), we obtain

$$F(t)F''(t) \geq \frac{q^-}{2}\left(F'(t)\right)^2, \quad \text{for all} \quad t \in [0, T]. \tag{6.11}$$

Setting $G(t) = F^{1-\frac{q^-}{2}}(t)$, we get

$$G'(t) = \left(1 - \frac{q^-}{2}\right)\frac{F'(t)}{F^{\frac{q^-}{2}}(t)}, \quad \text{and} \quad G''(t) = \left(1 - \frac{q^-}{2}\right)\frac{F(t)F''(t) - \frac{q^-}{2}\left(F'(t)\right)^2}{F^{1+\frac{q^-}{2}}(t)}.$$

Then we have $G''(t) \leq 0$, for all $t \in [0, T]$ due to (6.11). Thus, $G(t)$ is concave on $[0, T]$. This implies that

$$G(t) \leq G(t_0) + G'(t_0)(t - t_0), \quad \text{for all} \quad t \in [0, T]. \tag{6.12}$$

Replacing $t$ by $T$ in the above inequality and notice that (6.6), we obtain

$$\begin{aligned} G(T) &\leq G(t_0) + G'(t_0)(T - t_0) \\ &= F^{-\frac{q^-}{2}}(t_0)\left[F(t_0) - \left(\frac{q^-}{2} - 1\right)(T - t_0)F'(t_0)\right] \\ &= 0. \end{aligned}$$

This contradicts $G(T) > 0$, and the proof is complete.

## 7. Proof of Theorem 4.7

Assume that $u = u(t)$ is a global solution to (1.1). Then by virtue of (i) in Theorem 4.5, we get $J(u(t)) \geq 0$ for all $t \geq 0$. And therefore

$$\int_0^t \left\|u'(s)\right\|_2^2 \mathrm{d}s = J(u_0) - J(u(t)) \leq J(u_0).$$

Letting $t \to \infty$, one has

$$\int_0^\infty \left\|u'(s)\right\|_2^2 \mathrm{d}s \leq J(u_0) < \infty.$$

Therefore there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \left\|u'(t_n)\right\|_2 = 0, \tag{7.1}$$

which implies that

$$\left\|u'(t_n)\right\|_2 \leq A, \quad \forall n \in \mathbb{N},$$

for some a constant $A$.
Then we have

$$\begin{aligned} |I(u(t_n))| &= \left|\left\langle u'(t_n), u(t_n)\right\rangle\right| \\ &\leq \left\|u'(t_n)\right\|_2 \|u(t_n)\|_2 \\ &\leq \left\|u'(t_n)\right\|_2 S_2 \|\Delta u(t_n)\|_2 & (7.2) \\ &\leq AS_2 \|\Delta u(t_n)\|_2 , & (7.3) \end{aligned}$$

where $S_2$ is the constant given in (3.17).
Using the non-increasing property of $J(u(t))$, (7.3) and replacing $u$ by $u(t_n)$ in (3.2), we obtain

$$\begin{aligned} J(u_0) \geq J(u(t_n)) &\geq \left(\frac{1}{2} - \frac{1}{q^-}\right) \|\Delta u(t_n)\|_2^2 + \left(\frac{1}{p^+} - \frac{1}{q^-}\right) \int_\Omega |\nabla u(t_n)|^{p(x)} \mathrm{d}x + \frac{1}{q^-} I(u(t_n)) \\ &\geq \left(\frac{1}{2} - \frac{1}{q^-}\right) \|\Delta u(t_n)\|_2^2 - \frac{AS_2}{q^-} \|\Delta u(t_n)\|_2 , \end{aligned}$$

which implies that

$$\|\Delta u(t_n)\|_2 \leq \frac{AS_2 + \sqrt{A_2^2 S_2^2 + 2q^-(q^- - 2)J(u_0)}}{q^- - 2}. \tag{7.4}$$

The above inequality ensures that $\{u(t_n)\}$ is bounded in $H_0^2(\Omega)$. Then, since $H_0^2(\Omega)$ is reflexive, $H_0^2(\Omega) \hookrightarrow W_0^{1,p(.)}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ are compact embeddings (by (1.2) and (1.3)), there exists a sub-sequence of $\{u(t_n)\}$, still denoted by $\{u(t_n)\}$ and a $\phi \in H_0^2(\Omega)$ such that

$$u(t_n) \rightharpoonup \phi \quad \text{weakly in} \quad H_0^2(\Omega), \tag{7.5}$$

$$u(t_n) \to \phi \quad \text{strongly in} \quad W_0^{1,p(.)}(\Omega), \tag{7.6}$$

$$u(t_n) \to \phi \quad \text{strongly in} \quad L^{q(.)}(\Omega). \tag{7.7}$$

For any $v \in H_0^2(\Omega)$. Replacing $u$ by $u(t_n)$ in the equation (1.1), by multiplying (1.1) by $v$ and integrating by parts, we have

$$\begin{aligned} &\left|\langle \Delta u(t_n), \Delta v\rangle + \left\langle |\nabla u(t_n)|^{p(x)-2} \nabla u(t_n), \nabla v\right\rangle - \left\langle |u(t_n)|^{q(x)-2} u(t_n), v\right\rangle\right| \\ &\qquad = \left|\left\langle u'(t_n), v\right\rangle\right| \leq \left\|u'(t_n)\right\|_2 \|v\|_2 . \end{aligned}$$

From this and (7.1), it follows that

$$\lim_{n\to\infty}\left(\langle\Delta u(t_n),\Delta v\rangle+\left\langle|\nabla u(t_n)|^{p(x)-2}\nabla u(t_n),\nabla v\right\rangle-\left\langle|u(t_n)|^{q(x)-2}u(t_n),v\right\rangle\right)=0,$$

which, together with (7.5), (7.6) and (7.7) yields

$$\phi\in\mathcal{S}. \tag{7.8}$$

By (7.1), (7.2) and (7.4), we obtain

$$\lim_{n\to\infty}I(u(t_n))=0,$$

which, together with (7.6), (7.7) and (7.8), implies

$$\begin{aligned}\lim_{n\to\infty}\|\Delta u(t_n)\|_2&=-\lim_{n\to\infty}\int_\Omega|\nabla u(t_n)|^{p(x)}\,\mathrm{d}x+\lim_{n\to\infty}\int_\Omega|u(t_n)|^{q(x)}\,\mathrm{d}x\\&=-\int_\Omega|\nabla\phi|^{p(x)}\,\mathrm{d}x+\int_\Omega|\phi|^{q(x)}\,\mathrm{d}x=\|\Delta\phi\|_2\,.\end{aligned}\tag{7.9}$$

It is noticed that $H_0^2(\Omega)$ is uniformly convex. Then by (7.5) and (7.9), we imply (see [2, Proposition 3.32])

$$u(t_n)\to\phi\quad\text{strongly in}\quad H_0^2(\Omega).$$

The proof is complete.

## 8. Proof of Theorem 4.8, 4.9 and 4.10

By borrowing the ideas from [10, 29] we can prove the Theorem 4.8, 4.9 and 4.10 as follows.

*Proof of Theorem 4.8.* Assume that $u_0\in\mathcal{N}_+$ and $\|u_0\|_2\le\lambda_{J(u_0)}$. We first prove that $u(t)\in\mathcal{N}_+$ for all $t\in[0,T_{\max})$. Indeed, assume on the contrary that there is $t_0>0$ such that $u(t)\in\mathcal{N}_+$ for all $t\in[0,t_0)$ and $u(t_0)\in\mathcal{N}$. Then for all $t\in[0,t_0)$, we have

$$0<|I(u(t))|=\left|\langle u'(t),u(t)\rangle\right|\le\left\|u'(t)\right\|_2\|u(t)\|_2\,,$$

which gives $\|u'(t)\|_2>0$. From this and (4.2), we obtain $J(u(t_0))<J(u_0)$, i.e., $u(t_0)\in J^{J(u_0)}$. So $\|u(t_0)\|_2\ge\lambda_{J(u_0)}$. On the other hand, for all $t\in[0,t_0)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2=-2I\left(u(t)\right)<0,$$

which implies that $\|u(t_0)\|_2<\|u_0\|_2\le\lambda_{J(u_0)}$. We thus arrive at a contradiction and therefore it proves the claim $u(t)\in\mathcal{N}_+$ for all $t\in[0,T_{\max})$. This gives $u(t)\in\mathcal{N}_+\cap J^{J(u_0)}$ for all $t\in[0,T_{\max})$ by using the strictly decreasing property of $J(u(t))$. Then by virtue of (ii) in Lemma 3.5, $u(t)$ remains bounded in $H_0^2(\Omega)$ for all $t\in[0,T_{\max})$ so that $T_{\max}=\infty$, i.e., $u_0\in\mathcal{G}$. We next prove $u_0\in\mathcal{G}_0$. Let any $w\in\omega(u_0)$, we get

$$\|w\|_2<\lambda_{J(u_0)}\quad\text{and}\quad J(w)<J(u_0),$$

which implies that $\omega(u_0)\cap\mathcal{N}=\emptyset$ by definition of $\lambda_{J(u_0)}$. And therefore, $\omega(u_0)=\{0\}$, i.e., $u_0\in\mathcal{G}_0$.

Now we assume that $u_0\in\mathcal{N}_-$ and $\|u_0\|_2\ge\Lambda_{J(u_0)}$. By analogous arguments as above, we also have $u(t)\in\mathcal{N}_-$ for all $t\in[0,T_{\max})$. Assume on the contrary that $T_{\max}=\infty$, then for every $w\in\omega(u_0)$, one has

$$\|w\|_2>\Lambda_{J(u_0)}\quad\text{and}\quad J(w)<J(u_0),$$

which gives $\omega(u_0)\cap\mathcal{N}=\emptyset$ by the definition of $\Lambda_{J(u_0)}$. However, since $\operatorname{dist}(0,\mathcal{N}_-)>0$, we also have $0\notin\omega(u_0)$. And hence $\omega(u_0)=\emptyset$, which contradicts $T_{\max}=\infty$. Thus $u_0\in\mathcal{B}$. The proof is complete. $\square$

*Proof of Theorem 4.9.* Let any $u \in H_0^2(\Omega)\backslash\{0\}$. By using (3.2), we get

$$\begin{aligned} J(u) &\geq \left(\frac{1}{2}-\frac{1}{q^-}\right)\|\Delta u\|_2^2+\left(\frac{1}{p^+}-\frac{1}{q^-}\right)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x+\frac{1}{q^-}I(u) \\ &> \left(\frac{1}{2}-\frac{1}{q^-}\right)\|\Delta u\|_2^2+\frac{1}{q^-}I(u) \\ &\geq \left(\frac{1}{2}-\frac{1}{q^-}\right)S_2^{-2}\|u\|_2^2+\frac{1}{q^-}I(u). \end{aligned} \tag{8.1}$$

where $S_2$ is defined in (3.17).
Replacing $u$ by $u_0$ in (8.1) and using (4.5), we obtain

$$\begin{aligned} J(u_0) &> \left(\frac{1}{2}-\frac{1}{q^-}\right)S_2^{-2}\|u_0\|_2^2+\frac{1}{q^-}I(u_0) \\ &\geq J(u_0)+\frac{1}{q^-}I(u_0), \end{aligned}$$

which gives $I(u_0)<0$, i.e.,

$$u_0 \in \mathcal{N}_-. \tag{8.2}$$

For any $u \in \mathcal{N}_{J(u_0)}$, we have $I(u)=0$ and $J(u)<J(u_0)$. Then by using (8.1), we obtain

$$\|u\|_2^2 \leq \frac{2q^-S_2^2}{q^--2}J(u_0),$$

which, together with (4.5), implies

$$\|u\|_2 \leq \|u_0\|_2.$$

Taking supremum over $u \in N_{J(u_0)}$, we obtain

$$\Lambda_{J(u_0)} \leq \|u_0\|_2. \tag{8.3}$$

Then by virtue of Theorem 4.8, it follows from (8.2) and (8.3) that $u_0 \in \mathcal{N}_- \cap \mathcal{B}$. The proof is complete. ☐

*Proof of Theorem 4.10.* Let $M>d$ and assume that $\Omega_1$ and $\Omega_2$ are two arbitrarily disjoint open subdomains of $\Omega$. For any $\alpha>0$ and $v \in H_0^2(\Omega)\setminus\{0\}$. Then by virtue of Lemma 3.1 (i), we get $\lim\limits_{\alpha\to+\infty} J(\alpha v)=-\infty$. Hence, we can choose $v \in H_0^2(\Omega_1) \subset H_0^2(\Omega)$ and sufficiently large $\alpha$ such that

$$J(\alpha v)\leq 0 \text{ and } \|\alpha v\|_2^2 \geq \frac{2q^-S_2^2}{q^--2}M,$$

where $S_2$ is the constant given in (3.17).
We next pick a function $w \in H_0^2(\Omega_2) \subset H_0^2(\Omega)$ satisfying $J(w)=M-J(\alpha v)$. Then by setting $u_M := w+\alpha v$, we get $J(u_M)=J(w)+J(\alpha v)=M>d$ and

$$\|u_M\|_2^2 \geq \|\alpha v\|_2^2 \geq \frac{2q^-S_2^2}{q^--2}J(u_M).$$

And therefore, by applying Theorem 4.9, we imply that $u_M \in \mathcal{N}_- \cap \mathcal{B}$. The proof is complete. ☐

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