LOCAL WELL-POSEDNESS OF COMPRESSIBLE RADIATION HYDRODYNAMIC EQUATIONS WITH DENSITY-DEPENDENT VISCOSITIES AND VACUUM

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ABSTRACT. In this paper, we consider the Cauchy problem for three-dimensional isentropic compressible radiation hydrodynamic equations with density-dependent viscosity coefficients. When the viscosity coefficients are given as power of density (ρ^δ with $\delta > 1$), we establish the local-in-time existence of classical solutions containing a vacuum for large initial data. Here, we point out that the initial layer compatibility conditions are not necessary.

1. INTRODUCTION

It is well known that the radiation effects become remarkable in some regime when the temperature is high, e.g., in the high-temperature plasma physics [33] and the models of gaseous stars in astrophysics [16], so it is necessary to take the radiation effects into consideration in the classical hydrodynamics framework. For the physical background of radiation hydrodynamics, we refer to Pomraning [29] and Mihalas [27]. The couplings between fluid field and radiation field involve momentum source and energy source depending on the specific radiation intensity governed by the so-called photon (radiation) transfer equation. From a microscopic point of view, the radiation field is composed of photons. At any time t, six variables are required to specify the position of the photon in phase space, namely, three position variables and three momentum variables. In general, we denote the three position variables by the vector $x = (x_1, x_2, x_3)$. The three momentum variables are equivalently replaced by the frequency ν and the travel direction Ω of the photon. Then the distribution function $f = f(t, x, \nu, \Omega)$ can be defined such that

$$\mathrm{d}n = f(t, x, \nu, \Omega)\mathrm{d}x\mathrm{d}\nu\mathrm{d}\Omega,$$

where $\mathrm{d}n$ is the number of photons (at time t) at x, ν, Ω in the six-dimensional differential volume $\mathrm{d}x\mathrm{d}\nu\mathrm{d}\Omega$. It is conventional in radiative transfer context to introduce the specific radiation intensity $I = I(t, x, \nu, \Omega)$ instead of the distribution function $f(t, x, \nu, \Omega)$. The specific radiation intensity is defined by

$$I(t, x, \nu, \Omega) = ch\nu f(t, x, \nu, \Omega),$$

where c is the vacuum speed of light, h is Planck constant.

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We know that there are three basic interactions between photons and matter, namely emission, absorption, and scattering, then the radiation field can be described by a transport equation with a collision source term (see [29])

$$A_r = \sigma_e - \sigma_a I + \int_0^\infty \int_{S^2} \left(\frac{\nu}{\nu'}\sigma_s I' - \sigma_s' I\right) \mathrm{d}\Omega' \mathrm{d}\nu', \tag{1.1}$$

where $I = I(t,x,\nu,\Omega), I' = I(t,x,\nu',\Omega')$, $t \geq 0$ is the time, $x \in \mathbb{R}^3$ is the Euler spatial coordinate, $\nu, \nu' \geq 0$ are the frequency of photons, $\Omega, \Omega' \in S^2$ are the travel direction of photons, $S^2 \subset \mathbb{R}^3$ denotes the unit sphere in $\mathbb{R}^3$. $\sigma_e \geq 0$ is the rate of energy emission due to spontaneous process; $\sigma_a \geq 0$ denotes the absorption coefficient; σ_s is the "differential scattering coefficient" such that the probability of photon being scattered from ν' to ν contained in $d\nu$, from Ω' to Ω contained in $\mathrm{d}\Omega$, and traveling a distance $\mathrm{d}s$ is given by $\sigma_s(\nu' \to \nu, \Omega' \cdot \Omega)\mathrm{d}\nu\mathrm{d}\Omega\mathrm{d}s$. Therefore, the time rate of outscattering and inscattering within the volume element are

$$\text{outscattering} = \int_0^\infty \int_{S^2} \sigma_s(\nu \to \nu', \Omega \cdot \Omega') I(t,x,\nu,\Omega) \mathrm{d}\Omega' \mathrm{d}\nu',$$

$$\text{inscattering} = \int_0^\infty \int_{S^2} \sigma_s(\nu' \to \nu, \Omega' \cdot \Omega) I(t,x,\nu',\Omega') \mathrm{d}\Omega' \mathrm{d}\nu'.$$

Taking radiation contribution into account, we consider the following isentropic Navier-Stokes-Boltzmann equations:

$$\begin{cases} \dfrac{1}{c} I_t + \Omega \cdot \nabla I = A_r, \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ \left(\rho u + \dfrac{1}{c^2} F_r\right)_t + \operatorname{div}(\rho u \otimes u + P_r) + \nabla P_m = \operatorname{div} \mathbb{T}, \end{cases} \tag{1.2}$$

where the unknown functions $\rho(t,x), u(t,x) = \big(u^1(t,x), u^2(t,x), u^3(t,x)\big)$ represent the density and the velocity, respectively. P_m is the material pressure with the following equation of state for polytropic fluid:

$$P_m = A\rho^\gamma, \quad \gamma > 1,$$

where $A > 0$ is a constant. The viscous stress tensor $\mathbb{T}$ is given by

$$\mathbb{T} = \mu(\rho)(\nabla u + (\nabla u)^\top) + \lambda(\rho) \operatorname{div} u \, \mathbb{I}_3, \tag{1.3}$$

where $\mu(\rho)$ and $\lambda(\rho)$ are viscosity coefficients with the form

$$\mu(\rho) = \alpha\rho^\delta, \quad \lambda(\rho) = \beta\rho^\delta, \tag{1.4}$$

for some constant $\delta > 1$, and α and β are constants satisfying the physical constraints

$$\alpha > 0, \quad 2\alpha + 3\beta \geq 0.$$

F_r and P_r are the radiation flux and the radiation pressure tensor, respectively, which are defined by

$$\begin{cases} F_r = \displaystyle\int_0^\infty \int_{S^2} I(t,x,\nu,\Omega)\Omega \mathrm{d}\Omega \mathrm{d}\nu, \\ P_r = \displaystyle\int_0^\infty \int_{S^2} I(t,x,\nu,\Omega)\Omega \otimes \Omega \mathrm{d}\Omega \mathrm{d}\nu. \end{cases} \tag{1.5}$$

One of the motivation that we study the radiation system (1.2) lies on the fact that Navier-Stokes equations can be, to some extent, regarded as the non-radiation limit of the radiation Navier-Stoke-Boltzman system (1.2). In fact, from the "induced processes" and the local thermal equilibrium assumption, σ_e and σ_a can be written as

$$\begin{cases} \sigma_e(t,x,\nu,\rho) = K_a \overline{B}(\nu) \left(1 + \dfrac{c^2 I}{2h\nu^3}\right), \\ \sigma_a(t,x,\nu,\rho) = K_a \left(1 + \dfrac{c^2 \overline{B}(\nu)}{2h\nu^3}\right), \end{cases} \tag{1.6}$$

where $\overline{B}(\nu)$ is a function of ν which denotes the energy density of black body radiation, $K_a = K_a(t,x,\nu,\rho) \geq 0$. With this change, when $\sigma_s = 0$, system (1.2) can be reduced to

$$\begin{cases} \dfrac{1}{c} I_t + \Omega \cdot \nabla I = -K_a(I - \overline{B}(\nu)), \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P_m = \operatorname{div} \mathbb{T} + \dfrac{1}{c} \displaystyle\int_0^\infty \int_{S^2} K_a(I - \overline{B}(\nu))\Omega \mathrm{d}\Omega \mathrm{d}\nu, \end{cases} \tag{1.7}$$

from which we can see that the impact of radiation on dynamical properties of the fluid vanishes when $I = \overline{B}(\nu)$, i.e., the system serves as the Navier-Stokes equations. Actually, the rigorous justification of this limit was shown by Ducomet-Nečasová in [9], where they provided two different types of singular limits: one is the equilibrium diffusion limit which corresponds to the compressible Navier-Stokes equations with $I = \overline{B}(\nu)$; the other asymptotic regime corresponds to a non-equilibrium diffusion limit, where the radiation transfer equation is approximated by a diffusion equation.

Using (1.3)-(1.5) and $(1.2)_1$, system (1.2) can be rewritten as

$$\begin{cases} \dfrac{1}{c} I_t + \Omega \cdot \nabla I = A_r, \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P_m = -\rho^\delta L u + \nabla \rho^\delta \cdot S(u) - \dfrac{1}{c} \displaystyle\int_0^\infty \int_{S^2} A_r \Omega \mathrm{d}\Omega \mathrm{d}\nu, \end{cases} \tag{1.8}$$

where

$$\begin{cases} Lu = -\alpha \Delta u - (\alpha + \beta) \nabla \operatorname{div} u, \\ S(u) = \alpha(\nabla u + (\nabla u)^\top) + \beta \operatorname{div} u \, \mathbb{I}_3. \end{cases}$$

In this paper, we consider the Cauchy problem of (1.8) with the following initial data and far field behavior:

$$\begin{aligned} &(I,\rho,u)|_{t=0}=(I_0,\rho_0,u_0),\\ &(I,\rho,u)\to(0,0,0) \quad \text{as } |x|\to\infty, \quad t\geq 0, \quad (\nu,\Omega)\in\mathbb{R}^+\times S^2. \end{aligned} \tag{1.9}$$

We notice that vacuum can be allowed in our case.

For the compressible Navier-Stokes equations without radiation effects, the local existence and uniqueness of classical solutions has been obtained by Nash and Serrin [28, 30] in the absence of vacuum. Matsumura-Nishida [25, 26] studied the global existence of classical solutions under the condition that the initial data are suitably small. When the initial density need not to be positive and may vanish in some open sets, the global existence of weak solutions was first established by Lions in [23] for large initial data (see also Feireisl et al. [12]).

Due to the radiation effect, things become more complicated for both inviscid and viscous fluids. For the Euler-Boltzmann equations of inviscid fluids, Jiang-Zhong [34] obtained the local existence of C^1 solutions for Cauchy problem in multi-dimensional space. Jiang-Wang [15] proved that some C^1 solutions will blow up in finite time. Blanc-Ducomet [2] considered the Cauchy problem and showed the existence of global weak solutions. For the Navier-Stokes-Boltzmann equations of viscous fluids, Chen-Wang [4] proved the local existence of classical solutions for Cauchy problem away from vacuum. Ducomet-Nečasová obtained the large time behavior of strong solution in one dimensional space [8] and showed the global existence and uniqueness of smooth solution for small initial data near the radiative equilibrium [10]. We notice that the above results were obtained in the absence of vacuum. When vacuum appears, some difficulties arise, e.g., the degeneracy of the momentum equation. Li-Zhu studied the formulation of singularities to classical solutions with compactly supported density [19] and established the local well-posedness of strong solution containing a vacuum in homogeneous Sobolev space for general initial data satisfying initial compatibility conditions [20, 22] . They also investigated the existence and uniqueness of local regular solutions for Euler-Boltzmann equations [21]. We mention that the above results are all for the case of constant viscosity coefficients.

However, from a physical point of view, both the viscosity coefficients and the radiation coefficients may depend on the temperature [6, 7, 11]. For isentropic flow, this dependence on temperature is reduced to the dependence on density by Boyle and Gay-Lussac law. This is why we study the isentropic Navier-Stokes-Boltzmann equations with density dependent viscosities satisfying (1.4). As far as we know, there are fewer results for this case due to some essential difficulties. Wang [32] proved the local existence of strong solutions for the viscosity coefficients linearly depending on the density (i.e., $\delta=1$ in (1.4)). Our goal in this paper is to establish the local-in-time existence and uniqueness of classical solutions to (1.8)-(1.9) when the viscosity coefficients satisfy (1.4) with $\delta>1$, which has more singular structure and stronger degeneracy as well as stronger nonlinear source terms than the linear case.

The main difficulties and strategies in studying the local well-posedness of classical solutions can be summarized as follows.

- Compared with some other coupled systems of fluid models, for example, the MHD equations, the Navier-Stokes-Maxwell equations, and the Navier-Stokes-Poisson equations, etc., which are coupled with a parabolic or an elliptic equation, our system is a system of fluid equations coupled with a non-linear integro-differential hyperbolic equation, which makes things more complicated. According to the coupling between the density, the velocity and the radiation intensity, we need to find a suitable linearized structure, which is the very first step towards the proof of existence result.
- The vacuum leads to double degeneracy: the degeneracy of the time evolution of velocity and the degeneracy of viscosity in momentum equation $(1.8)_2$. It is well known that it is difficult to determine and control the velocity near vacuum, which is the key point in the vacuum related problems. Recently, for the degenerate Navier-Stokes equations, Li-Pan-Zhu [18] successfully controlled the behavior of velocity by introducing some new variables and making use of the "quasi-symmetric hyperbolic"−"degenerate elliptic" structure, which can be expected to be generalized to the radiation version in this paper.
- Difficulties also arise due to the non-local terms, the nonlinear terms, and the coupled cross terms between radiation field and fluid field, which bring us some trouble in obtaining the uniform a *priori* estimates. We shall make full use of the reformulated structure with the aid of some physically reasonable assumptions on the radiation coefficients.
- The time continuity of higher order derivative is not trivial. We need to establish the time weighted energy estimates of $t^{\frac{1}{2}}u_t \in L^\infty(0,T;H^2)$ by using the reformulated structure and Lemma 2.6. It should be pointed out that our work is carried out in the inhomogeneous Sobolev space, and the initial layer compatibility conditions are no longer needed.
- We also have some technical difficulties, for example, since the Poincaré type inequality will no longer be valid for the Cauchy problem because of the unboundedness of the domain, we need to adopt some other techniques, such as the Gagliardo-Nirenberg inequality and the Sobolev interpolation inequality instead. For the estimates of radiation terms, it is not easy to get due to the complexity of the multiple integrals, we also need to use some tools like Minkowski's inequality, etc.

The rest of this paper is organized as follows. In §2, we collect some notations and elementary inequalities, introduce some reasonable assumptions on the radiation coefficients, and the main result is stated in the end of this section. In §3, after introducing some new variables and the "quasi-symmetric hyperbolic"−"degenerate elliptic" structure, we reformulate the original problem (1.8)-(1.9) into (3.1) and establish the corresponding local existence result. The main theorem, Theorem 2.1, is proved in §4.

2. PRELIMINARIES AND MAIN RESULT

In this section, we first give some notations and some elementary inequalities which will be used throughout this paper, and then we introduce some assumptions on the radiation coefficients and state our main result.

2.1. **Notations and elementary inequalities.** We first give some notations.

(1) The following notations are adopted for the homogeneous and inhomogeneous Sobolev spaces (see [13]).

$$
\begin{aligned}
&|f|_p = \|f\|_{L^p(\mathbb{R}^3)}\,, \quad \|f\|_s = \|f\|_{H^s(\mathbb{R}^3)}\,,\\
&D^{k,r} = \left\{ f \in L^1_{loc}(\mathbb{R}^3) : |\nabla^k f|_r < +\infty \right\},\\
&D^k = D^{k,2}, \quad |f|_{D^k} = \|f\|_{D^k(\mathbb{R}^3)} = |\nabla^k f|_2, \quad k \geq 2,\\
&D^1 = \left\{ f \in L^6(\mathbb{R}^3) : |f|_{D^1} = \|f\|_{D^1(\mathbb{R}^3)} = |\nabla f|_2 < +\infty \right\}.
\end{aligned}
$$

(2) For simplicity, we also use the following notations:

$$
\begin{aligned}
&\|(f,g)\|_X := \|f\|_X + \|g\|_X\,,\\
&\|f(t,x,\nu,\Omega)\|_{X_1(\mathbb{R}^+\times S^2; X_2([0,T]\times\mathbb{R}^3))} := \Big\| \|f(\nu,\Omega,\cdot,\cdot)\|_{X_2([0,T]\times\mathbb{R}^3)} \Big\|_{X_1(\mathbb{R}^+\times S^2)}\,.
\end{aligned}
$$

The following well-known Gagliardo-Nirenberg inequality and Sobolev interpolation inequality can be found in [17, 24].

Lemma 2.1. [17] *For $p \in [2,6], q \in (1,+\infty)$ and $r \in (3,+\infty)$, there exists a constant $C > 0$ depending only on q, r, such that for any $f \in D^1$ and $g \in L^q \cap D^{1,r}$, we have*

$$
\begin{aligned}
&|f|_p \leq C\,|f|_2^{\frac{6-p}{2p}}\,|\nabla f|_2^{\frac{3p-6}{2p}}\,,\\
&|g|_\infty \leq C\,|g|_q^{\frac{q(r-3)}{3r+q(r-3)}}\,|\nabla g|_r^{\frac{3r}{3r+q(r-3)}}\,.
\end{aligned}
$$

Lemma 2.2. [24] *For $s' \in [0,s]$, there exists a constant $C_s > 0$ depending on s, such that for any $u \in H^s$, we have*

$$
\|u\|_{s'} \leq C_s \|u\|_0^{1-\frac{s'}{s}} \|u\|_s^{\frac{s'}{s}}.
$$

The following Minskowski's inequality will be used in the estimates of the radiation terms.

Lemma 2.3 (Minkowski's inequality [1])**.** *For $1 \leq p \leq q \leq +\infty$, we have*

$$
\Big\| \|f(\cdot,x_2)\|_{L^p(X_1)} \Big\|_{L^q(X_2)} \leq \Big\| \|f(x_1,\cdot)\|_{L^q(X_2)} \Big\|_{L^p(X_1)}.
$$

The following lemma is used to get the compactness information of solutions.

Lemma 2.4 (Aubin-Lions Lemma [31])**.** *Let $X_0 \subset X \subset X_1$ be three Banach spaces. Suppose that X_0 is compactly embedded in X and X is continuously embedded in X_1. Then the following statements hold.*

(1) *If J is bounded in $L^p([0,T]; X_0)$ for $1 \leq p < +\infty$, and $\frac{\partial J}{\partial t}$ is bounded in $L^1([0,T]; X_1)$, then J is relatively compact in $L^p([0,T]; X)$;*
(2) *If J is bounded in $L^\infty([0,T]; X_0)$ and $\frac{\partial J}{\partial t}$ is bounded in $L^p([0,T]; X_1)$ for $p > 1$, then J is relatively compact in $C([0,T]; X)$.*

The following Moser-type calculus inequalities can be found in Majda [24].

Lemma 2.5. [24] *Let r, a and b be constants such that*

$$\frac{1}{r}=\frac{1}{a}+\frac{1}{b}, \quad \text{and} \quad 1\leq a,\ b,\ r\leq +\infty.$$

$\forall s\geq 1$, if $f,g\in W^{s,a}\cap W^{s,b}(\mathbb{R}^3)$, then it holds

$$\begin{aligned} &|\nabla^s(fg)-f\nabla^s g|_r\leq C_s\left(|\nabla f|_a|\nabla^{s-1}g|_b+|\nabla^s f|_b|g|_a\right),\\ &|\nabla^s(fg)-f\nabla^s g|_r\leq C_s\left(|\nabla f|_a|\nabla^{s-1}g|_b+|\nabla^s f|_a|g|_b\right), \end{aligned} \tag{2.1}$$

where $C_s>0$ is a constant depending only on s, and $\nabla^s f$ $(s>1)$ is the set of all $\partial_x^\zeta f$ with $|\zeta|=s$. Here $\zeta=(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{R}^3$ is a multi-index.

The following lemma is used to obtain the time weighted estimates like $t^{\frac{1}{2}}u_t\in L^\infty(0,T;H^2)$, etc.

Lemma 2.6. [3] *If $f(t,x)\in L^2([0,T];L^2)$, then there exists a sequence s_k such that*

$$s_k\to 0, \quad \text{and} \quad s_k|f(s_k,x)|_2^2\to 0, \quad \text{as} \quad k\to+\infty.$$

2.2. **Hypothesis on the radiation quantities.** The general form of radiation coefficients is usually not known, since it is difficult to evaluate these physical coefficients in quantum mechanics. Physically speaking, the radiation coefficients can be written as $\sigma_e=\rho\overline{\sigma}_e, \sigma_a=\rho\overline{\sigma}_a, \sigma_s=\rho\overline{\sigma}_s$, and then A_r can be written as $A_r=\rho\overline{A}_r$, where

$$\overline{A}_r=\overline{\sigma}_e-\overline{\sigma}_a I+\int_0^\infty\int_{S^2}\Big(\frac{\nu}{\nu'}\overline{\sigma}_s I'-\overline{\sigma}'_s I\Big)\,\mathrm{d}\Omega'\mathrm{d}\nu'. \tag{2.2}$$

Next, we will introduce some physically reasonable assumptions on the radiation coefficients $\overline{\sigma}_e,\overline{\sigma}_a$, and $\overline{\sigma}_s$ which are similar as in [20, 32, 34].

H1(**Differential scattering coefficient**) Let $\sigma_s=\overline{\sigma}_s(\nu'\to\nu,\Omega'\cdot\Omega)\rho=\rho\overline{\sigma}_s$ and $\sigma'_s=\overline{\sigma}'_s(\nu\to\nu',\Omega\cdot\Omega')\rho=\rho\overline{\sigma}'_s$, where the functions $\overline{\sigma}_s\geq 0$ and $\overline{\sigma}'_s\geq 0$ satisfy

$$\begin{cases} \displaystyle\int_0^\infty\int_{S^2}\Big(\int_0^\infty\int_{S^2}\Big|\frac{\nu}{\nu'}\Big|^2|\overline{\sigma}_s|^2\,\mathrm{d}\Omega'\mathrm{d}\nu'\Big)^{\lambda_1}\mathrm{d}\Omega\mathrm{d}\nu\leq\alpha,\\ \displaystyle\int_0^\infty\int_{S^2}\Big(\int_0^\infty\int_{S^2}\overline{\sigma}'_s\mathrm{d}\Omega'\mathrm{d}\nu'\Big)^{\lambda_2}\mathrm{d}\Omega\mathrm{d}\nu+\int_0^\infty\int_{S^2}\overline{\sigma}'_s\mathrm{d}\Omega'\mathrm{d}\nu'\leq\alpha, \end{cases} \tag{2.3}$$

where $\lambda_1=1$ or $1/2$, $\lambda_2=1$ or 2, and $\alpha>0$ is a fixed constant.

H2(**Emission and absorption coefficients**) Let $\sigma_a=\sigma_a(t,x,\nu,\Omega,\rho)=\rho\overline{\sigma}_a(t,x,\nu,\Omega,\rho)$ and $\sigma_e=\sigma_e(t,x,\nu,\Omega,\rho)=\rho\overline{\sigma}_e(t,x,\nu,\Omega,\rho)$, where $\overline{\sigma}_e\geq 0,\overline{\sigma}_a\geq 0$. Let $\varphi=\rho^{\frac{\delta-1}{2}}$, then we assume that for $s=0,1,2,3$,

$$\begin{cases} \|\overline{\sigma}_e\|_{L^1\cap L^2(\mathbb{R}^+\times S^2;C([0,T];H^s))}\leq M(\|\varphi\|_3)(1+\|\varphi\|_s),\\ \|(\overline{\sigma}_e)_t\|_{L^1\cap L^2(\mathbb{R}^+\times S^2;C([0,T];H^2))}\leq M(\|\varphi\|_3)(1+\|\varphi_t\|_2),\\ \|\overline{\sigma}_a\|_{L^\infty\cap L^2(\mathbb{R}^+\times S^2;C([0,T];H^s))}\leq M(\|\varphi\|_3)(1+\|\varphi\|_s),\\ \|(\overline{\sigma}_a)_t\|_{L^\infty\cap L^2(\mathbb{R}^+\times S^2;C([0,T];H^2))}\leq M(\|\varphi\|_3)(1+\|\varphi_t\|_2), \end{cases} \tag{2.4}$$

and

$$\begin{cases} \left\|\overline{\sigma}_e(\cdot,\cdot,\cdot,\cdot,\varphi^1)-\overline{\sigma}_e(\cdot,\cdot,\cdot,\cdot,\varphi^2)\right\|_{L^1\cap L^2(\mathbb{R}^+\times S^2;C([0,T];L^2))} \\ \qquad \le M\left(\left\|(\varphi^1,\varphi^2)\right\|_3\right)(1+|\varphi^1-\varphi^2|_2), \\ \left\|\overline{\sigma}_a(\cdot,\cdot,\cdot,\cdot,\varphi^1)-\overline{\sigma}_a(\cdot,\cdot,\cdot,\cdot,\varphi^2)\right\|_{L^\infty\cap L^2(\mathbb{R}^+\times S^2;C([0,T];L^2))} \\ \qquad \le M\left(\left\|(\varphi^1,\varphi^2)\right\|_3\right)(1+|\varphi^1-\varphi^2|_2), \end{cases} \tag{2.5}$$

where $M=M(\cdot)$ denotes a strictly increasing function from $[0,+\infty)$ to $[1,+\infty)$.

Remark 2.1. *We point out that we do have some physical models with radiation coefficients satisfying these assumptions. For instance (see* [21, 29, 34]*),*

(i) *A particular expression of absorption coefficient σ_a is given as*

$$\sigma_a(t,x,\nu,\Omega,\rho,\theta)=C_1\rho\theta^{-\frac{1}{2}}\exp\left(-\frac{C_2}{\theta^{\frac{1}{2}}}\left(\frac{\nu-\nu_0}{\nu_0}\right)^2\right),$$

where θ is the temperature, ν_0 is the fixed frequency, and $C_i (i=1,2)$ are positive constants. For the polytropic gas, i.e., $P_m=R\rho\theta=A\rho^\gamma$, where A,R are positive constants. Then we know that $\rho^{\frac{\gamma-1}{2}}=C_3\theta^{\frac{1}{2}}$ with $C_3=(\sqrt{A})^{-1}\sqrt{R}$, and

$$\begin{aligned}\sigma_a(t,x,\nu,\Omega,\rho)&=C_1C_3\rho\rho^{\frac{1-\gamma}{2}}\exp\left(-C_2C_3\rho^{\frac{1-\gamma}{2}}\left(\frac{\nu-\nu_0}{\nu_0}\right)^2\right)\\&=\rho\overline{\sigma}_a(t,x,\nu,\Omega,\rho),\end{aligned}$$

saitisfying

$$\begin{aligned}\lim_{\rho\to0}\frac{\sigma_a(t,x,\nu,\Omega,\rho)}{\rho}&=\lim_{\rho\to0}\overline{\sigma}_a(t,x,\nu,\Omega,\rho)=0,\\ \lim_{\rho\to+\infty}\frac{\sigma_a(t,x,\nu,\Omega,\rho)}{\rho}&=\lim_{\rho\to+\infty}\overline{\sigma}_a(t,x,\nu,\Omega,\rho)=0.\end{aligned}$$

(ii) *The Compton scattering kernel is given by*

$$\begin{aligned}\sigma_s(\nu\to\nu',\xi)=&\frac{C_4\rho(1+\xi^2)}{[1+C_5\nu(1-\xi)]^2}\left(1+\frac{C_5^2\nu^2(1-\xi)^2}{(1+\xi^2)[1+C_5\nu(1-\xi)]}\right)\\&\cdot\delta\left(\nu'-\frac{\nu}{1+C_5\nu(1-\xi)}\right),\\=&\rho\overline{\sigma}_s(\nu\to\nu',\xi),\end{aligned}$$

where $\xi=\Omega\cdot\Omega'$, $C_i (i=4,5)$ are positive constants, and $\delta(\cdot)$ is the delta function.

It is not difficult to verify that the σ_a and σ_s in the above models satisfy the assumptions **H1** and **H2**.

Remark 2.2. *From* (2.4) *and* (2.5)*, one can easily know that for $s=0,1,2,3$,*

$$\begin{cases}\|\sigma_e\|_{L^1\cap L^2(\mathbb{R}^+\times S^2;C([0,T];H^s))}\le M(\|\varphi\|_3)\,\|\varphi\|_3^{\frac{2}{\delta-1}}\,(1+\|\varphi\|_s),\\ \|\sigma_a\|_{L^\infty\cap L^2(\mathbb{R}^+\times S^2;C([0,T];H^s))}\le M(\|\varphi\|_3)\,\|\varphi\|_3^{\frac{2}{\delta-1}}\,(1+\|\varphi\|_s),\end{cases} \tag{2.6}$$

and

$$\begin{cases} \left\|\sigma_e(\cdot,\cdot,\cdot,\cdot,\varphi^1)-\sigma_e(\cdot,\cdot,\cdot,\cdot,\varphi^2)\right\|_{L^1\cap L^2(\mathbb{R}^+\times S^2;C([0,T];L^2))} \\ \qquad \le M\left(\left\|(\varphi^1,\varphi^2)\right\|_3\right)\|\varphi\|_3^{\frac{2}{\delta-1}}(1+|\varphi^1-\varphi^2|_2), \\ \left\|\sigma_a(\cdot,\cdot,\cdot,\cdot,\varphi^1)-\sigma_a(\cdot,\cdot,\cdot,\cdot,\varphi^2)\right\|_{L^\infty\cap L^2(\mathbb{R}^+\times S^2;C([0,T];L^2))} \\ \qquad \le M\left(\left\|(\varphi^1,\varphi^2)\right\|_3\right)\|\varphi\|_3^{\frac{2}{\delta-1}}(1+|\varphi^1-\varphi^2|_2). \end{cases} \tag{2.7}$$

2.3. **Main result.** Before stating the main result, let us first give the definition of regular solutions.

Definition 2.1 (Regular solutions)**.** *Let $T>0$ be a finite constant. (I,ρ,u) is called a regular solution to* (1.8)-(1.9) *if*

(1) (I,ρ,u) *satisfies the Cauchy problem* (1.8)-(1.9) *in the sense of distribution*;

(2) $I\in L^2(\mathbb{R}^+\times S^2;C([0,T];H^3)),\quad I_t\in L^2(\mathbb{R}^+\times S^2;C([0,T];H^2))$;

(3) $\rho\ge 0,\quad \rho^{\frac{\delta-1}{2}}\in C([0,T];H^3),\quad (\rho^{\frac{\delta-1}{2}})_t\in C([0,T];H^2)$;

(4) $\rho^{\frac{\gamma-1}{2}}\in C([0,T];H^3),\quad (\rho^{\frac{\gamma-1}{2}})_t\in C([0,T];H^2)$;

(5) $u\in C([0,T];H^{s'})\cap L^\infty([0,T];H^3),\quad \rho^{\frac{\delta-1}{2}}\nabla^4 u\in L^2([0,T];L^2)$,
$u_t\in C([0,T];H^1)\cap L^2([0,T];D^2)$;

(6) $u_t+u\cdot\nabla u=-\dfrac{1}{c}\displaystyle\int_0^\infty\int_{S^2}\left(\lim_{\rho\to 0}\overline{A}_r\right)\Omega d\Omega d\nu,$ *when* $\rho(t,x)=0$,

for some constant $s'\in[2,3)$.

We are now in position to state our main result.

Theorem 2.1. *Let δ,γ be positive such that*

$$1<\delta\le\min\{\gamma,5/3\}. \tag{2.8}$$

Assume that the initial data (I_0,ρ_0,u_0) satisfy

$$I_0\in L^2(\mathbb{R}^+\times S^2;H^3),\quad \rho_0\ge 0,\quad \left(\rho_0^{\frac{\delta-1}{2}},\rho_0^{\frac{\gamma-1}{2}},u_0\right)\in H^3, \tag{2.9}$$

then there exists a unique regular solution (I,ρ,u) to (1.8)-(1.9) *in $(t,x,\nu,\Omega)\in[0,T_*]\times\mathbb{R}^3\times\mathbb{R}^+\times S^2$. Moreover, (I,ρ,u) is also a classical solution for $t\in(0,T_*]$.*

Remark 2.3. *We point out that our range of parameter δ is smaller than the corresponding case for isentropic Navier-Stokes equations in* [14]*. This is caused by the appearance of the radiation terms.*

Remark 2.4. *We can find the following class of initial data (I_0,ρ_0,u_0) satisfying the condition* (2.9)*:*

$$I_0\in L^2(\mathbb{R}^+\times S^2;C_0^3(\mathbb{R}^3)),\quad \rho_0(x)=\frac{x}{1+|x|^{2m}},\quad u_0\in C_0^3(\mathbb{R}^3),$$

where $m>\max\left\{\dfrac{1}{2},\dfrac{2+\delta}{2(\delta-1)},\dfrac{2+\gamma}{2(\gamma-1)}\right\}$.

3. REFORMULATION AND LOCAL-IN-TIME WELL-POSEDNESS

As we said in the introduction, we first need to reformulate our problem to figure out the intrinsic special structure of the system and to prove the well-posedness of strong solutions to the reformulated problem.

By introducing new variables $\varphi = \rho^{\frac{\delta-1}{2}}$ and $\phi = \frac{2\sqrt{A\gamma}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}$, problem (1.8)-(1.9) can be rewritten as

$$\begin{cases} \frac{1}{c}I_t + \Omega\cdot\nabla I = A_r, \\ \varphi_t + u\cdot\nabla\varphi + \frac{\delta-1}{2}\varphi\operatorname{div}u = 0, \\ \phi_t + u\cdot\nabla\phi + \frac{\gamma-1}{2}\phi\operatorname{div}u = 0, \\ u_t + u\cdot\nabla u + \frac{\gamma-1}{2}\phi\nabla\phi + \varphi^2 Lu = \nabla\varphi^2\cdot Q(u) - \frac{1}{c}\int_0^\infty\int_{S^2}\overline{A}_r\Omega \mathrm{d}\Omega\mathrm{d}\nu, \\ (I,\varphi,\phi,u)|_{t=0} = (I_0,\varphi_0,\phi_0,u_0), \\ (I,\varphi,\phi,u)\to(0,0,0,0), \quad \text{as} \quad |x|\to+\infty, \quad t\geq 0, \end{cases} \tag{3.1}$$

where $(t,x,\nu,\Omega)\in\mathbb{R}^+\times\mathbb{R}^3\times\mathbb{R}^+\times S^2$, $\varphi_0 = \rho_0^{\frac{\delta-1}{2}}$, $\phi_0 = \frac{2\sqrt{A\gamma}}{\gamma-1}\rho_0^{\frac{\gamma-1}{2}}$, $Q(u)$ is given by

$$Q(u) = \frac{\delta}{\delta-1}S(u),$$

and $\overline{A}_r$ is defined in (2.2). Letting $U = (\phi,u)^\top$, problem (3.1) can be reduced to

$$\begin{cases} \frac{1}{c}I_t + \Omega\cdot\nabla I = A_r, \\ \varphi_t + u\cdot\nabla\varphi + \frac{\delta-1}{2}\varphi\operatorname{div}u = 0, \\ U_t + \sum_{j=1}^3 A_j(U)U_{x_j} = -\varphi^2F(U) + G(\varphi,U) + H(I,\varphi), \\ (I,\varphi,U)|_{t=0} = (I_0,\varphi_0,U_0), \\ (I,\varphi,U)\to(0,0,0), \quad \text{as} \quad |x|\to+\infty, \quad t\geq 0, \end{cases} \tag{3.2}$$

where $U_0 = (\phi_0,u_0)^\top$,

$$A_j(U) = \begin{pmatrix} u^j & \frac{\gamma-1}{2}\phi e_j \\ \frac{\gamma-1}{2}\phi e_j^\top & u^j\,\mathbb{I}_3 \end{pmatrix}, \quad j=1,2,3, \tag{3.3}$$

and $e_j = (\delta_{1j},\delta_{2j},\delta_{3j})$ is the Kronecker symbol with $\delta_{ij}=1, i=j$ and $\delta_{ij}=0, i\neq j$. It is easily noticed that $A_j(j=1,2,3)$ are all symmetric matrices. The source terms F,G,H

are respectively given by

$$\begin{aligned} & F(U)=\begin{pmatrix} 0 \\ Lu \end{pmatrix}, \quad G(\varphi,U)=\begin{pmatrix} 0 \\ \nabla\varphi^2\cdot Q(u) \end{pmatrix}, \\ & H(I,\varphi)=\left(0,-\frac{1}{c}\int_0^\infty\int_{S^2}\overline{A}_r\Omega \mathrm{d}\Omega \mathrm{d}\nu\right)^\top . \end{aligned} \tag{3.4}$$

The following is concerned with the local existence and uniqueness of strong solutions to Cauchy problem (3.2).

Theorem 3.1. *Let* (2.8) *hold. Assume that the initial data* $(I_0,\varphi_0,\phi_0,u_0)$ *satisfy*

$$I_0\in L^2(\mathbb{R}^+\times S^2;H^3),\quad \varphi_0,\phi_0\geq 0,\quad (\varphi_0,\phi_0,u_0)\in H^3. \tag{3.5}$$

Then there exists a unique strong solution (I,φ,ϕ,u) *to* (3.2) *in* $(0,T_*]\times\mathbb{R}^3\times\mathbb{R}^+\times S^2$ *satisfying*

$$\begin{aligned} & I\in L^2(\mathbb{R}^+\times S^2;C([0,T_*];H^3)),\quad I_t\in L^2(\mathbb{R}^+\times S^2;C([0,T_*];H^2)),\\ & \varphi\in C([0,T_*];H^3),\quad \varphi_t\in C([0,T_*];H^2),\\ & \phi\in C([0,T_*];H^3),\quad \phi_t\in C([0,T_*];H^2),\\ & u\in C([0,T_*];H^{s'})\cap L^\infty([0,T_*];H^3),\quad \varphi\nabla^4u\in L^2([0,T_*];L^2),\\ & u_t\in C([0,T_*];H^1)\cap L^2([0,T_*];D^2), \end{aligned} \tag{3.6}$$

for any constant $s'\in[2,3)$.

The proof of Theorem 3.1 can be divided into the following steps:

- Linearization. First, we linearize the original nonlinear problem (3.2) and add an artificial viscosity η^2Lu to remove the degeneracy of viscosity of the original problem. The linearized problem (3.7) is solved in Lemma 3.1 by using standard arguments.
- Uniform estimates and vanish viscosity limit. Then, we derive the uniform *a priori* estimates which are independent of the artificial viscosity coefficient η, based on which, the corresponding degenerate linearized problem (3.45) can be solved by taking vanishing viscosity limit.
- Iteration and convergence. Finally, we construct the suitable approximate solutions by using the linearized problem (3.45) and prove the convergence to obtain the strong solution to the original nonlinear problem (3.2).

3.1. **Linearized problem with an artificial viscosity.** In order to solve the nonlinear problem (3.2), we first consider the following linearized problem (3.7) with an artificial viscosity, and then establish the corresponding uniform *a priori* estimates.

3.1.1. *Linearization and solvability.* The linearized problem to be studied reads as:

$$\begin{cases} \frac{1}{c} I_t + \Omega \cdot \nabla I = \widetilde{A}_r, \\ \varphi_t + v \cdot \nabla \varphi + \frac{\delta - 1}{2} \omega \operatorname{div} v = 0, \\ U_t + \sum_{j=1}^{3} A_j(V) U_{x_j} = -(\varphi^2 + \eta^2) F(U) + G(\varphi, V) + H(I, \varphi), \\ (I, \varphi, U)|_{t=0} = (I_0, \varphi_0, U_0), \\ (I, \varphi, U) \to (0,0,0), \quad \text{as} \quad |x| \to +\infty, \quad t \geq 0, \end{cases} \tag{3.7}$$

where

$$\widetilde{A}_r = \sigma_e - \sigma_a I + \int_0^\infty \int_{S^2} \left(\frac{\nu}{\nu'} \sigma_s \chi - \sigma_s' I \right) \mathrm{d}\Omega' \mathrm{d}\nu',$$

$V = (\psi, v)^\top$, (χ, ω, ψ) are known functions and $v = (v^1, v^2, v^3)$ is a known vector, satisfying $(\chi, \omega, V)(0, x) = (I_0, \varphi_0, U_0)$ with $U_0 = (\phi_0, u_0)^\top$ and

$$\begin{aligned} & \chi \in L^2(\mathbb{R}^+ \times S^2; C([0,T]; H^3)), \quad \chi_t \in L^2(\mathbb{R}^+ \times S^2; C([0,T]; H^2)), \\ & (\omega, \psi) \in C([0,T]; H^3), \quad (\omega_t, \psi_t) \in C([0,T]; H^2), \\ & v \in C([0,T]; H^{s'}) \cap L^\infty([0,T]; H^3), \quad \omega \nabla^4 v \in L^2([0,T]; L^2), \\ & v_t \in C([0,T]; H^1) \cap L^2([0,T]; D^2), \end{aligned} \tag{3.8}$$

for any constant $s' \in [2, 3)$.

The global well-posedness of strong solutions (I, φ, ϕ, u) to (3.7) can be obtained as below by using standard arguments ([5, 17]) for any fixed $\eta > 0$.

Lemma 3.1. *Assume that the initial data satisfy* (3.5). *Then for any fixed $\eta > 0$ and $T > 0$, there exists a unique strong solution (I, φ, ϕ, u) to* (3.7) *satisfying*

$$\begin{aligned} & I \in L^2(\mathbb{R}^+ \times S^2; C([0,T]; H^3)), \quad I_t \in L^2(\mathbb{R}^+ \times S^2; C([0,T]; H^2)), \\ & (\varphi, \phi) \in C([0,T]; H^3), \quad (\varphi_t, \phi_t) \in C([0,T]; H^2), \\ & u \in C([0,T]; H^{s'}) \cap L^\infty([0,T]; H^3), \quad \varphi \nabla^4 u \in L^2([0,T]; L^2), \\ & u_t \in C([0,T]; H^1) \cap L^2([0,T]; D^2), \end{aligned} \tag{3.9}$$

for any constant $s' \in [2, 3)$.

3.1.2. *Uniform a priori estimates of solutions.* Let (I, φ, ϕ, u) be a strong solution to (3.7) in $(0, T] \times \mathbb{R}^3 \times \mathbb{R}^+ \times S^2$, with initial data satisfying (3.5). We will establish the uniform *a priori* estimates for (I, φ, ϕ, u), which are independent of η. First, we choose a positive constant c_0 independent of η such that

$$2 + |(\varphi_0, \phi_0)|_\infty + \|(\varphi_0, \phi_0)\|_3 + \|I_0\|_{L^2(\mathbb{R}^+ \times S^2; H^3)} + \|u_0\|_3 \leq c_0. \tag{3.10}$$

Then we assume that there exist $T^* \in (0, T]$ and positive constants $c_i (i = 1, \cdots, 5)$ such that

$$1 < c_0 \leq c_1 \leq c_2 \leq c_3 \leq c_4 \leq c_5,$$

and

$$
\begin{aligned}
\sup_{0\leq t\leq T^*}\|\chi\|_{L^2(\mathbb{R}^+\times S^2;H^3)}\leq c_1,\ &\sup_{0\leq t\leq T^*}\|\chi_t\|_{L^2(\mathbb{R}^+\times S^2;H^2)}\leq c_2,\\
\sup_{0\leq t\leq T^*}\left(\|(\omega,\psi)\|_2^2+\|v\|_2^2\right)&+\int_0^{T^*}|\omega\nabla^3 v|_2^2\mathrm{d}t\leq c_3^2,\\
\sup_{0\leq t\leq T^*}\left(|(\omega,\psi)|_{D^3}^2+|v|_{D^3}^2\right)&+\int_0^{T^*}|\omega\nabla^4 v|_2^2\mathrm{d}t\leq c_4^2,\\
\sup_{0\leq t\leq T^*}\left(\|(\omega_t,\psi_t)\|_2^2+\|v_t\|_1^2\right)&+\int_0^{T^*}|v_t|_{D^1\cap D^2}^2\mathrm{d}t\leq c_5^2,
\end{aligned}
\tag{3.11}
$$

where T^* and c_i $(i=1,...,5)$ depend only on c_0 and the fixed constants $A,\alpha,\beta,\gamma,\delta,T$, and will be determined later. Hereinafter, we use $C\geq 1$ to denote a generic constant, which may change in different places.

Now we state some *a priori* estimates.

Lemma 3.2. *Let (I,φ,U) be the unique strong solution to* (3.7). *Then there exists a time T_1 such that*

$$
\begin{aligned}
&1+|\varphi|_\infty^2+\|\varphi\|_3^2\leq Cc_0^2,\\
&\|\varphi_t\|_1^2\leq Cc_3^4,\quad |\varphi_t|_{D^2}^2\leq Cc_4^4,
\end{aligned}
\tag{3.12}
$$

for $0\leq t\leq T_1=\min\{T^*,(1+c_4)^{-2}\}$.

Proof. According to $(3.7)_2$ and using the standard energy estimates, we deduce that

$$
\begin{aligned}
\|\varphi\|_3\leq& C\left(\|\varphi_0\|_3+\int_0^t(\|\omega\|_3\|v\|_3+|\omega\nabla^4 v|_2)\mathrm{d}s\right)\exp\left(C\int_0^t\|v\|_3\mathrm{d}s\right)\\
\leq& C\left(c_0+c_4^2t+t^{\frac{1}{2}}\left(\int_0^t|\omega\nabla^4 v|_2^2\mathrm{d}s\right)^{1/2}\right)\exp(Cc_4t)\\
\leq& Cc_0,
\end{aligned}
\tag{3.13}
$$

for $0\leq t\leq T_1=\min\{T^*,(1+c_4)^{-2}\}$. It follows from $(3.7)_2$ that for $0\leq t\leq T_1$

$$
\begin{aligned}
|\varphi_t|_2\leq& C(|v|_6|\nabla\varphi|_3+|\omega|_\infty|\nabla v|_2)\leq Cc_3^2,\\
|\varphi_t|_{D^1}\leq& C(|\nabla v|_6|\nabla\varphi|_3+|v|_\infty|\nabla^2\varphi|_2+|\nabla\omega|_3|\nabla v|_6+|\omega|_\infty|\nabla^2 v|_2)\leq Cc_3^2,\\
|\varphi_t|_{D^2}\leq& C|v\cdot\varphi+\psi\operatorname{div}v|_{D^2}\leq C(\|v\|_2\|\nabla\varphi\|_2+\|\omega\|_2\|\nabla v\|_2)\leq Cc_4^2.
\end{aligned}
$$

□

Lemma 3.3. *Let (I,φ,U) be the unique strong solution to* (3.7). *Then there exists a time T_2 such that*

$$
\begin{aligned}
&\|I\|_{L^2(\mathbb{R}^+\times S^2;C([0,T_2];H^3))}\leq Cc_0,\\
&\|I_t\|_{L^2(\mathbb{R}^+\times S^2;C([0,T_2];H^2))}\leq CM(c_0)c_0^{l+1}c_1,
\end{aligned}
\tag{3.14}
$$

for $0\leq t\leq T_2=\min\{T_1,(1+M(c_0)c_0^lc_1)^{-2}\}$ *and* $l=2/(\delta-1)$.

Proof. Multiplying $(3.7)_1$ by $2cI$ and integrating over $\mathbb{R}^3$, we conclude that

$$\begin{aligned}\frac{d}{dt}|I|_2^2 \leq & C\left(|I|_2|\sigma_e|_2 - \int\left(\sigma_a + \int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}\nu'\right)|I|^2\mathrm{d}x\right.\\ &\left. + |I|_2|\varphi|_\infty^l\|\chi\|_{L^2(\mathbb{R}^+\times S^2;L^2)}\left(\int_0^\infty\int_{S^2}\left|\frac{\nu}{\nu'}\right|^2|\overline{\sigma}_s|^2\,\mathrm{d}\Omega'\mathrm{d}\nu'\right)^{1/2}\right)\\ \leq & C\left(|I|_2^2 + |\sigma_e|_2^2 + c_0^{2l}c_1^2\int_0^\infty\int_{S^2}\left|\frac{\nu}{\nu'}\right|^2|\overline{\sigma}_s|^2\,\mathrm{d}\Omega'\mathrm{d}\nu'\right),\end{aligned} \tag{3.15}$$

where we have used the fact that $\sigma_a \geq 0$ and $\sigma_s' \geq 0$.

Applying the operator $\partial_x^\zeta(1 \leq |\zeta| \leq 3)$ to $(3.7)_1$, we have

$$\begin{aligned}&\frac{1}{c}\partial_x^\zeta I_t + \Omega\cdot\nabla\partial_x^\zeta I + \left(\sigma_a + \int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}\nu'\right)\partial_x^\zeta I\\ =& -\left(\partial_x^\zeta(\sigma_a I) - \sigma_a\partial_x^\zeta I\right) - \int_0^\infty\int_{S^2}\overline{\sigma}_s'\left(\partial_x^\zeta(\varphi^l I) - \varphi^l\partial_x^\zeta I\right)\mathrm{d}\Omega'\mathrm{d}\nu'\\ &+ \partial_x^\zeta\sigma_e + \int_0^\infty\int_{S^2}\frac{\nu}{\nu'}\overline{\sigma}_s\partial_x^\zeta(\varphi^l\chi)\mathrm{d}\Omega'\mathrm{d}\nu'.\end{aligned} \tag{3.16}$$

Multiplying (3.16) by $2\partial_x^\zeta I(1 \leq |\zeta| \leq 3)$ and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}&\frac{1}{c}\frac{d}{dt}\int|\partial_x^\zeta I|^2\mathrm{d}x + \int\left(\sigma_a + \int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}\nu'\right)(\partial_x^\zeta I)^2\mathrm{d}x\\ =& -\int\left(\partial_x^\zeta(\sigma_a I) - \sigma_a\partial_x^\zeta I\right)\partial_x^\zeta I\mathrm{d}x\\ &-\int\int_0^\infty\int_{S^2}\overline{\sigma}_s'\left(\partial_x^\zeta(\varphi^l I) - \varphi^l\partial_x^\zeta I\right)\mathrm{d}\Omega'\mathrm{d}\nu'\partial_x^\zeta I\mathrm{d}x\\ &+\int\partial_x^\zeta\sigma_e\partial_x^\zeta I\mathrm{d}x + \int\left(\int_0^\infty\int_{S^2}\frac{\nu}{\nu'}\overline{\sigma}_s\partial_x^\zeta(\varphi^l\chi)\mathrm{d}\Omega'\mathrm{d}\nu'\right)\partial_x^\zeta I\mathrm{d}x\\ \triangleq&\sum_{i=1}^4 R_i,\end{aligned} \tag{3.17}$$

with the natural correspondence for $R_i\,(i = 1,2,3,4)$, which can be treated one by one as follows. Using Hölder's inequality, Gagliardo-Nirenberg inequality and Lemma 2.5, we estimate $R_i(i = 1,2,3,4)$ as follows:

$$
\begin{aligned}
R_1 \le & C|\nabla\sigma_a|_\infty|I|_2|\nabla I|_2, \quad \text{if } |\zeta|=1,\\
R_1 \le & C(|\nabla^2\sigma_a|_2|I|_\infty+|\nabla\sigma_a|_3|\nabla I|_6)|\nabla^2 I|_2, \quad \text{if } |\zeta|=2,\\
R_1 \le & C(|\nabla^3\sigma_a|_2|I|_\infty+|\nabla\sigma_a|_\infty|\nabla^2 I|_2)|\nabla^3 I|_2, \quad \text{if } |\zeta|=3,\\
R_2 \le & C|\varphi^l|_3|I|_6|\nabla I|_2, \quad \text{if } |\zeta|=1,\\
R_2 \le & C(|\nabla^2\varphi^l|_2|I|_\infty+|\nabla\varphi^l|_3|\nabla I|_6)|\nabla^2 I|_2, \quad \text{if } |\zeta|=2,\\
R_2 \le & C(|\nabla^3\varphi^l|_2|I|_\infty+|\nabla\varphi^l|_\infty|\nabla^2 I|_2)|\nabla^3 I|_2, \quad \text{if } |\zeta|=3,\\
R_3 \le & C(|\nabla\sigma_e|_2^2+|\nabla I|_2^2), \quad \text{if } |\zeta|=1,\\
R_3 \le & C(|\nabla^2\sigma_e|_2^2+|\nabla^2 I|_2^2), \quad \text{if } |\zeta|=2,\\
R_3 \le & C(|\nabla^3\sigma_e|_2^2+|\nabla^3 I|_2^2), \quad \text{if } |\zeta|=3,\\
R_4 \le & C|\partial_x^\zeta I|_2\int_0^\infty\int_{S^2}\frac{\nu}{\nu'}\overline{\sigma}_s(|\varphi|_\infty^{l-1}|\nabla\varphi|_6|\chi|_3+|\varphi|_\infty^l|\nabla\chi|_2)\mathrm{d}\Omega'\mathrm{d}\nu'\\
\le & C|\nabla I|_2\|\varphi\|_2^l\|\chi\|_{L^2(\mathbb{R}^+\times S^2;H^1)}\left(\int_0^\infty\int_{S^2}\left|\frac{\nu}{\nu'}\right|^2|\overline{\sigma}_s|^2\,\mathrm{d}\Omega'\mathrm{d}\nu'\right)^{1/2}\\
\le & C\left(|\nabla I|_2^2+c_0^{2l}c_1^2\int_0^\infty\int_{S^2}\left|\frac{\nu}{\nu'}\right|^2|\overline{\sigma}_s|^2\,\mathrm{d}\Omega'\mathrm{d}\nu'\right), \quad \text{if } |\zeta|=1,
\end{aligned}
$$

$$
\begin{aligned}
R_4 \le & C|\partial_x^\zeta I|_2\int_0^\infty\int_{S^2}\frac{\nu}{\nu'}\overline{\sigma}_s(|\nabla^2\varphi^l|_2|\chi|_\infty+|\nabla\varphi^l|_6|\nabla\chi|_3+|\varphi|_\infty^l|\nabla^2\chi|_2)\mathrm{d}\Omega'\mathrm{d}\nu'\\
\le & C|\nabla^2 I|_2\|\varphi^l\|_2\|\chi\|_{L^2(\mathbb{R}^+\times S^2;H^2)}\left(\int_0^\infty\int_{S^2}\left|\frac{\nu}{\nu'}\right|^2|\overline{\sigma}_s|^2\,\mathrm{d}\Omega'\mathrm{d}\nu'\right)^{1/2}\\
\le & C\left(|\nabla^2 I|_2^2+c_0^{2l}c_1^2\int_0^\infty\int_{S^2}\left|\frac{\nu}{\nu'}\right|^2|\overline{\sigma}_s|^2\,\mathrm{d}\Omega'\mathrm{d}\nu'\right), \quad \text{if } |\zeta|=2,\\
R_4 \le & C|\partial_x^\zeta I|_2\Bigg(\int_0^\infty\int_{S^2}\frac{\nu}{\nu'}\overline{\sigma}_s(|\nabla^3\varphi^l|_2|\chi|_\infty+|\nabla^2\varphi^l|_6|\nabla\chi|_3)\mathrm{d}\Omega'\mathrm{d}\nu'\\
& \qquad +\int_0^\infty\int_{S^2}\frac{\nu}{\nu'}\overline{\sigma}_s(|\nabla\varphi^l|_3|\nabla^2\chi|_6+|\varphi|_\infty^l|\nabla^3\chi|_2)\mathrm{d}\Omega'\mathrm{d}\nu'|\Bigg)\\
\le & C|\nabla^3 I|_2\|\varphi^l\|_3\|\chi\|_{L^2(\mathbb{R}^+\times S^2;H^3)}\left(\int_0^\infty\int_{S^2}\left|\frac{\nu}{\nu'}\right|^2|\overline{\sigma}_s|^2\,\mathrm{d}\Omega'\mathrm{d}\nu'\right)^{1/2}\\
\le & C\left(|\nabla^3 I|_2^2+c_0^{2l}c_1^2\int_0^\infty\int_{S^2}\left|\frac{\nu}{\nu'}\right|^2|\overline{\sigma}_s|^2\,\mathrm{d}\Omega'\mathrm{d}\nu'\right), \quad \text{if } |\zeta|=3.
\end{aligned}
$$

Plugging the above estimates for R_i $(i=1,2,3,4)$ into (3.17) and summing up all $\zeta(1\le|\zeta|\le 3)$, combining with (3.15), and using (2.6)-(2.7), we then obtain

$$
\begin{aligned}
\frac{d}{dt}\|I\|_3^2 \leq & C\Bigg(\left(\|\sigma_a\|_{L^\infty(\mathbb{R}^+\times S^2;H^3)} + \|\varphi^l\|_3\right)\|I\|_3^2 + \|\sigma_e\|_3^2 \\
& \qquad + c_0^{2l}c_1^2 \int_0^\infty \int_{S^2} \left|\frac{\nu}{\nu'}\right|^2 |\overline{\sigma}_s|^2 \,\mathrm{d}\Omega'\mathrm{d}\nu' \Bigg) \\
\leq & C\left(M(c_0)c_0^{l+1}\|I\|_3^2 + \|\sigma_e\|_3^2 + c_0^{2l}c_1^2 \int_0^\infty \int_{S^2} \left|\frac{\nu}{\nu'}\right|^2 |\overline{\sigma}_s|^2 \,\mathrm{d}\Omega'\mathrm{d}\nu' \right).
\end{aligned} \tag{3.18}
$$

Integrating (3.18) over $\mathbb{R}^+ \times S^2$ and using Gronwall's inequality, we conclude that

$$
\begin{aligned}
& \|I\|^2_{L^2(\mathbb{R}^+\times S^2;C([0,T_2];H^3))} \\
\leq & C\left(\left(\|I_0\|^2_{L^2(\mathbb{R}^+\times S^2;H^3)} + M^2(c_0)c_0^{2(l+1)}T_2 + c_0^{2l}c_1^2T_2\right)\exp\left(CM(c_0)c_0^{l+1}T_2\right)\right) \\
\leq & Cc_0^2,
\end{aligned} \tag{3.19}
$$

for $0 \leq t \leq T_2 = \min\{T_1, (1 + M(c_0)c_0^l c_1)^{-2}\}$.

According to the equation $(3.7)_1$, for $0 \leq t \leq T_2$, we arrive at

$$
\begin{aligned}
& \|I_t\|_{L^2(\mathbb{R}^+\times S^2;C([0,T_2];H^2))} \\
\leq & C\Bigg(\|I\|_{L^2(\mathbb{R}^+\times S^2;C([0,T_2];H^3))} + \|\sigma_e\|_{L^2(\mathbb{R}^+\times S^2;C([0,T_2];H^2))} \\
& + \|\sigma_a\|_{L^2(\mathbb{R}^+\times S^2;C([0,T_2];H^2))}\|I\|_{L^2(\mathbb{R}^+\times S^2;C([0,T_2];H^2))} \\
& + \|\varphi^l\|_2\|\chi\|^2_{L^2(\mathbb{R}^+\times S^2;H^2)}\left(\int_0^\infty \int_{S^2} \left|\frac{\nu}{\nu'}\right|^2 |\overline{\sigma}_s|^2 \,\mathrm{d}\Omega'\mathrm{d}\nu'\right)^{1/2} \\
& + \|\varphi^l\|_2\|I\|_{L^2(\mathbb{R}^+\times S^2;C([0,T_2];H^2))}\int_0^\infty \int_{S^2} \overline{\sigma}'_s \mathrm{d}\Omega'\mathrm{d}\nu' \Bigg) \\
\leq & CM(c_0)c_0^{l+1}c_1.
\end{aligned} \tag{3.20}
$$

□

Next, we will establish the a *prior*i estimates for U.

Lemma 3.4. *Let (I, φ, U) be the unique strong solution to* (3.7)*. Then there exists a time T_3 such that*

$$
\begin{aligned}
& \|U\|_2^2 + \int_0^t |\sqrt{\varphi^2+\eta^2}\nabla^3 u|_2^2 ds \leq Cc_0^2, \\
& |U_t|_2^2 + |\phi_t|^2_{D^1} + \int_0^t |u_t|^2_{D^1}\, ds \leq CM^2(c_0)c_3^6,
\end{aligned} \tag{3.21}
$$

for $0 \leq t \leq T_3 = \min\{T_2, (1 + M(c_0)c_0^l c_4)^{-4})\}$.

Proof. We denote $U_{x_j}=\partial_j U$. Applying the operator ∂_x^ζ to $(3.7)_3$, we obtain

$$
\begin{aligned}
&\partial_x^\zeta U_t+\sum_{j=1}^3 A_j(V)\partial_j\partial_x^\zeta U+(\varphi^2+\eta^2)F(\partial_x^\zeta U)\\
=&-\sum_{j=1}^3\Big(\partial_x^\zeta(A_j(V)\partial_j U)-A_j(V)\partial_j\partial_x^\zeta U\Big)-\Big(\partial_x^\zeta\left((\varphi^2+\eta^2)F(U)\right)-(\varphi^2+\eta^2)F(\partial_x^\zeta U)\Big)\\
&+\nabla\varphi^2\cdot Q(\partial_x^\zeta v)+\Big(\partial_x^\zeta(\nabla\varphi^2\cdot Q(v))-\nabla\varphi^2\cdot Q(\partial_x^\zeta v)\Big)\\
&-\frac{1}{c}\int_0^\infty\int_{S^2}\partial_x^\zeta(\overline{A}_r)\Omega\mathrm{d}\Omega\mathrm{d}\nu.
\end{aligned}
\tag{3.22}
$$

Multiplying (3.22) by $2\partial_x^\zeta U(|\zeta|\le 2)$ and integrating over $\mathbb{R}^3$, it is shown that

$$
\begin{aligned}
&\frac{d}{dt}|\partial_x^\zeta U|_2^2+2\alpha|\sqrt{\varphi^2+\eta^2}\nabla(\partial_x^\zeta u)|_2^2+2(\alpha+\beta)|\sqrt{\varphi^2+\eta^2}\operatorname{div}(\partial_x^\zeta u)|_2^2\\
=&\int(\partial_x^\zeta U)^\top\operatorname{div}A(V)\partial_x^\zeta U\mathrm{d}x-2\sum_{j=1}^3\int\Big(\partial_x^\zeta(A_j(V)\partial_j U)-A_j(V)\partial_j\partial_x^\zeta U\Big)\partial_x^\zeta U\mathrm{d}x\\
&+2\int\nabla(\varphi^2+\eta^2)\cdot\Big(\alpha\nabla(\partial_x^\zeta u)+(\alpha+\beta)\operatorname{div}(\partial_x^\zeta u)\mathbb{I}\Big)\cdot\partial_x^\zeta u\mathrm{d}x\\
&-2\int\Big(\partial_x^\zeta((\varphi^2+\eta^2)Lu)-(\varphi^2+\eta^2)L(\partial_x^\zeta u)\Big)\cdot\partial_x^\zeta u\mathrm{d}x\\
&+2\int\nabla\varphi^2\cdot Q(\partial_x^\zeta v)\cdot\partial_x^\zeta u dx+2\int\Big(\partial_x^\zeta(\nabla\varphi^2\cdot Q(v))-\nabla\varphi^2\cdot Q(\partial_x^\zeta v)\Big)\cdot\partial_x^\zeta u\mathrm{d}x\\
&-\frac{2}{c}\int\int_0^\infty\int_{S^2}\partial_x^\zeta(\overline{A}_r)\Omega\cdot\partial_x^\zeta u\mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x\\
\triangleq&\sum_{i=1}^7 J_i,
\end{aligned}
\tag{3.23}
$$

with the natural correspondence for $J_i\,(i=1,2,\cdots,7)$ and $\operatorname{div}A(V)=\sum_{j=1}^3(A_j)_{x_j}$.

Now we first estimate $J_i\,(i=1,2,\cdots,6)$ one by one. It follows from the Gagliardo-Nirenberg inequality, Hölder's inequality and Young's inequality that

$$
\begin{aligned}
J_1\le&C|\nabla V|_\infty|\partial_x^\zeta U|_2^2\le Cc_4|\partial_x^\zeta U|_2^2,\\
J_2\le&C|\partial_x^\zeta(A_j(V)\partial_j U)-A_j(V)\partial_j\partial_x^\zeta U|_2|\partial_x^\zeta U|_2\le C|\nabla V|_\infty|\nabla U|_2^2\le Cc_4|\nabla U|_2^2,\quad\text{if } |\zeta|\le 1,\\
J_2\le&C(|\nabla V|_\infty|\nabla^2 U|_2+|\nabla^2 V|_3|\nabla U|_6)|\nabla^2 U|_2\le Cc_4|\nabla^2 U|_2^2,\quad\text{if } |\zeta|=2,\\
J_3\le&C|\varphi\nabla(\partial_x^\zeta u)|_2|\nabla\varphi|_\infty|\partial_x^\zeta u|_2\le\frac{\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla(\partial_x^\zeta u)|_2^2+Cc_0^2|\partial_x^\zeta u|_2^2,\\
J_4\le&C|\nabla\varphi|_\infty|\varphi Lu|_2|\partial_x^\zeta u|_2\le\frac{\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla^2 u|_2^2+Cc_0^2|\partial_x^\zeta u|_2^2,\quad\text{if } |\zeta|\le 1,
\end{aligned}
$$

$$
\begin{aligned}
J_4 \le & C\int(|\nabla^2\varphi^2 Lu|+|\nabla\varphi^2\nabla Lu|)|\nabla^2 u|\mathrm{d}x\\
\le & C\left(|\varphi\nabla^2 u|_6|\nabla^2\varphi|_3|Lu|_2+|\nabla\varphi|_\infty^2|Lu|_2|\nabla^2 u|_2+|\varphi\nabla^3 u|_2|\nabla^2 u|_2\right)\\
\le & C\left(|\nabla(\varphi\nabla^2 u)|_2|\nabla^2\varphi|_3|Lu|_2+|\nabla\varphi|_\infty^2|Lu|_2|\nabla^2 u|_2+|\varphi\nabla^3 u|_2|\nabla^2 u|_2\right)\\
\le & C\left(\left(|\nabla\varphi|_\infty|\nabla^2 u|_2+|\varphi\nabla^3 u|_2\right)|\nabla^2\varphi|_3|Lu|_2+\left(|\nabla\varphi|_\infty^2|Lu|_2+|\varphi\nabla^3 u|_2\right)|\nabla^2 u|_2\right)\\
\le & \frac{\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla^3 u|_2^2+Cc_0^2|\nabla^2 u|_2^2,\quad \text{if } |\zeta|=2,\\
J_5 \le & C|\varphi|_\infty|\nabla\varphi|_\infty\|\nabla v\|_1|\partial_x^\zeta u|_2\le Cc_3^3|\partial_x^\zeta u|_2,\\
J_6 \le & C(|\varphi|_\infty|\nabla^2\varphi|_2|\nabla v|_\infty+|\nabla\varphi|_\infty^2|\nabla v|_2)|\nabla u|_2\le Cc_4^3|\nabla u|_2,\quad \text{if } |\zeta|\le 1,\\
J_6 \le & C\big(|\varphi|_\infty|\nabla^3\varphi|_2|\nabla v|_\infty+|\nabla^2\varphi|_2|\nabla\varphi|_\infty|\nabla v|_\infty\\
&+|\varphi|_\infty|\nabla^2\varphi|_3|\nabla^2 v|_6+|\nabla\varphi|_\infty^2|\nabla^2 v|_2|\nabla^2 u|_2\big)\le Cc_4^3|\nabla^2 u|_2,\quad \text{if } |\zeta|=2.
\end{aligned}
$$

For J_7, using the definition of $\overline{A}_r$, we have

$$
\begin{aligned}
J_7=&-\frac{1}{c}\int\int_0^\infty\int_{S^2}\partial_x^\zeta\overline{\sigma}_e\partial_x^\zeta u\cdot\Omega\mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x\\
&+\frac{1}{c}\int\int_0^\infty\int_{S^2}\left(\overline{\sigma}_a+\int_0^\infty\int_{S^2}\overline{\sigma}_s'\mathrm{d}\Omega'\mathrm{d}\nu'\right)\partial_x^\zeta I\partial_x^\zeta u\cdot\Omega\mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x\\
&+\int\int_0^\infty\int_{S^2}\left(\partial_x^\zeta(\overline{\sigma}_a I)-\overline{\sigma}_a\partial_x^\zeta I\right)\partial_x^\zeta u\cdot\Omega\mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x\\
&-\frac{1}{c}\int\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\frac{\nu}{\nu'}\overline{\sigma}_s\partial_x^\zeta I'\mathrm{d}\Omega'\mathrm{d}\nu'\partial_x^\zeta u\cdot\Omega\mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x\triangleq\sum_{i=1}^4 J_{7i},
\end{aligned}\tag{3.24}
$$

with the natural correspondence for $J_{7i}(i=1,2,3,4)$ which are estimated as follows.

$$
\begin{aligned}
J_{71}\le & C\|\partial_x^\zeta\overline{\sigma}_e\|_{L^1(\mathbb{R}^+\times S^2;L^2)}|\partial_x^\zeta u|_2\le CM(c_0)c_0|\partial_x^\zeta u|_2,\\
J_{72}\le & C|\partial_x^\zeta u|_2\int_0^\infty\int_{S^2}\left(|\overline{\sigma}_a|_{L^\infty}+\int_0^\infty\int_{S^2}\overline{\sigma}_s'\mathrm{d}\Omega'\mathrm{d}\nu'\right)|\partial_x^\zeta I|_2\mathrm{d}\Omega\mathrm{d}\nu\\
\le & C|\partial_x^\zeta u|_2\left(\|\overline{\sigma}_a\|_{L^2(\mathbb{R}^+\times S^2;L^\infty)}+1\right)\|\partial_x^\zeta I\|_{L^2(\mathbb{R}^+\times S^2;L^2)}\le CM(c_0)c_0^2|\partial_x^\zeta u|_2,\\
J_{73}\le & C|\partial_x^\zeta u|_2\|\nabla\overline{\sigma}_a\|_{L^2(\mathbb{R}^+\times S^2;L^\infty)}\|I\|_{L^2(\mathbb{R}^+\times S^2;L^2)}\le CM(c_0)c_0^2|\partial_x^\zeta u|_2,\quad \text{if } |\zeta|\le 1,\\
J_{73}\le & C|\partial_x^\zeta u|_2\Big(\|\nabla^2\overline{\sigma}_a\|_{L^2(\mathbb{R}^+\times S^2;L^2)}\|I\|_{L^2(\mathbb{R}^+\times S^2;L^2)}\\
&+\|\nabla\overline{\sigma}_a\|_{L^2(\mathbb{R}^+\times S^2;L^3)}\|\nabla I\|_{L^2(\mathbb{R}^+\times S^2;L^6)}\Big)\\
\le & CM(c_0)c_0^2|\partial_x^\zeta u|_2,\quad \text{if } |\zeta|=2,\\
J_{74}\le & C|\partial_x^\zeta u|_2\|\partial_x^\zeta I\|_{L^2(\mathbb{R}^+\times S^2;L^2)}\le Cc_0|\partial_x^\zeta u|_2,
\end{aligned}
$$

where we used the Hypothesis **H1** and **H2**.

Substituting $J_i(i=1,\cdots,7)$ into (3.23) and summing up all $|\zeta|\le 2$, we obtain

$$
\frac{d}{dt}\|U\|_2^2+|\sqrt{\varphi^2+\eta^2}\nabla^3 u|_2^2\le C\left(M^2(c_0)c_4^2\|U\|_2^2+c_4^4\right),\tag{3.25}
$$

which, along with the Gronwall's inequality, yields

$$\|U\|_2^2+\int_0^t|\sqrt{\varphi^2+\eta^2}\nabla^3 u|_2^2\mathrm{d}s\leq C\left(c_0^2+c_4^4 t\right)\exp\left(CM^2(c_0)c_4^2 t\right)\leq Cc_0^2, \tag{3.26}$$

for $0\leq t\leq T_3=\min\{T_2,(1+M(c_0)c_0^l c_4)^{-4}\}$.

For $|\partial_x^\zeta U_t|_2$ with $|\zeta|\leq 1$, according to the equation $(3.7)_3$, one has

$$\begin{aligned}
&|\phi_t|_2\leq C(|v|_\infty|\nabla\phi|_2+|\psi|_\infty|\nabla u|_2)\leq Cc_3^2,\\
&|\phi_t|_{D^1}\leq C(|\nabla v|_\infty|\nabla\phi|_2+|v|_\infty|\nabla^2\phi|_2+|\psi|_\infty|\nabla^2 u|_2+|\nabla\psi|_3|\nabla u|_6)\leq Cc_3^2,\\
&|u_t|_2=\left|-v\cdot\nabla u-\frac{\gamma-1}{2}\psi\nabla\phi-(\varphi^2+\eta^2)Lu+\nabla\varphi^2\cdot Q(v)-\frac{1}{c}\int_0^\infty\int_{S^2}\overline{A}_r\Omega\mathrm{d}\Omega\mathrm{d}\nu\right|_2\\
&\qquad\leq C\Big(\left(|v|_6|\nabla u|_3+|\psi|_\infty|\nabla\phi|_2+|\varphi^2+\eta^2|_\infty|\nabla^2 u|_2+|\varphi|_\infty|\nabla\varphi|_\infty|\nabla v|_2\right)\\
&\qquad\qquad+\|\overline{\sigma}_e\|_{L^1(\mathbb{R}^+\times S^2;L^2)}+\left(\|\overline{\sigma}_a\|_{L^2(\mathbb{R}^+\times S^2;L^\infty)}+1\right)\|I\|_{L^2(\mathbb{R}^+\times S^2;L^2)}\Big)\\
&\qquad\leq CM(c_0)c_3^3.
\end{aligned}\tag{3.27}$$

Similarly, we have

$$\begin{aligned}
&|u_t|_{D^1}=\left|-v\cdot\nabla u-\frac{\gamma-1}{2}\psi\nabla\phi-(\varphi^2+\eta^2)Lu+\nabla\varphi^2\cdot Q(v)-\frac{1}{c}\int_0^\infty\int_{S^2}\overline{A}_r\Omega\mathrm{d}\Omega\mathrm{d}\nu\right|_{D^1}\\
\leq&C\Big(|\nabla v|_3|\nabla u|_6+|v|_\infty|\nabla^3 u|_2+|\psi|_\infty|\nabla^2\phi|_2+|\nabla\psi|_\infty|\nabla\phi|_2+|\nabla\varphi|_\infty|\varphi|_\infty|u|_{D^2}\\
&+|\sqrt{\varphi^2+\eta^2}|_\infty|\sqrt{\varphi^2+\eta^2}\nabla^3 u|_2+(|\varphi|_\infty^2+\|\nabla\varphi\|_2)\|\nabla v\|_1+\|\nabla\overline{\sigma}_e\|_{L^1(\mathbb{R}^+\times S^2;L^\infty)}\\
&+\|\nabla\overline{\sigma}_a\|_{L^2(\mathbb{R}^+\times S^2;L^\infty)}\|I\|_{L^2(\mathbb{R}^+\times S^2;L^2)}+\left(\|\overline{\sigma}_a\|_{L^2(\mathbb{R}^+\times S^2;L^\infty)}+1\right)\|\nabla I\|_{L^2(\mathbb{R}^+\times S^2;L^2)}\Big)\\
\leq&C\left(M(c_0)c_3^2+c_0|\sqrt{\varphi^2+\eta^2}\nabla^3 u|_2\right),
\end{aligned}\tag{3.28}$$

which implies that

$$\int_0^t|u_t|_{D^1}^2\mathrm{d}s\leq C\int_0^t\left(M^2(c_0)c_3^4+c_0^2|\sqrt{\varphi^2+\eta^2}\nabla^3 u|_2^2\right)\mathrm{d}s\leq Cc_0^4, \tag{3.29}$$

for $0\leq t\leq T_3$. □

Lemma 3.5. *Let (I,φ,U) be the unique strong solution to (3.7). Then we have*

$$\begin{aligned}
&|U|_{D^3}^2+\int_0^t|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 ds\leq Cc_0^2,\\
&|u_t|_{D^1}^2+|\phi_t|_{D^2}^2+\int_0^t|u_t|_{D^2}^2 ds\leq CM^2(c_0)c_4^4,
\end{aligned}\tag{3.30}$$

for $0\leq t\leq T_3$.

Proof. From the proof in Lemma 3.4, for $|\zeta|=3$, we also have

$$
\begin{aligned}
&\frac{d}{dt}|\partial_x^\zeta U|_2^2+2\alpha|\sqrt{\varphi^2+\eta^2}\nabla(\partial_x^\zeta u)|_2^2+2(\alpha+\beta)|\sqrt{\varphi^2+\eta^2}\operatorname{div}(\partial_x^\zeta u)|_2^2\\
=&\int(\partial_x^\zeta U)^\top \operatorname{div} A(V)\partial_x^\zeta U\mathrm{d}x-2\sum_{j=1}^3\int\Big(\partial_x^\zeta(A_j(V)\partial_j U)-A_j(V)\partial_j\partial_x^\zeta U\Big)\partial_x^\zeta U\mathrm{d}x\\
&+2\int\nabla(\varphi^2+\eta^2)\cdot\Big(\alpha\nabla(\partial_x^\zeta u)+(\alpha+\beta)\operatorname{div}(\partial_x^\zeta u)\mathbb{I}_3\Big)\cdot\partial_x^\zeta u\mathrm{d}x\\
&-2\int\Big(\partial_x^\zeta((\varphi^2+\eta^2)Lu)-(\varphi^2+\eta^2)L(\partial_x^\zeta u)\Big)\cdot\partial_x^\zeta u\mathrm{d}x\\
&+2\int\nabla\varphi^2\cdot Q(\partial_x^\zeta v)\cdot\partial_x^\zeta u dx+2\int\Big(\partial_x^\zeta(\nabla\varphi^2\cdot Q(v))-\nabla\varphi^2\cdot Q(\partial_x^\zeta v)\Big)\cdot\partial_x^\zeta u\mathrm{d}x\\
&-\frac{2}{c}\int\int_0^\infty\int_{S^2}\partial_x^\zeta(\overline{A}_r)\Omega\cdot\partial_x^\zeta u\mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x\triangleq\sum_{i=8}^{14}J_i,
\end{aligned}\tag{3.31}
$$

with the natural correspondence for $J_i(i=8,9,\cdots,14)$ which can be treated one by one as follows. Using Hölder's inequality, Young's inequality and Lemma 2.5, we first estimate $J_i(i=8,9,\cdots,12)$ as follows.

$$
\begin{aligned}
J_8\le&C|\nabla V|_\infty|\nabla^3 u|_2^2\le Cc_4|\nabla^3 u|_2^2,\\
J_9\le&C|\partial_x^\zeta(A_j(V)\partial_j U)-A_j(V)\partial_j\partial_x^\zeta U|_2|\partial_x^\zeta U|_2\\
\le&C(|\nabla^3 V|_2\|\nabla U\|_2+|\nabla V|_\infty|\nabla U|_2)|\nabla^3 U|_2\le C(c_4|\nabla^3 U|_2^2+c_4^3),\\
J_{10}\le&C|\varphi\nabla(\partial_x^\zeta u)|_2|\nabla\varphi|_\infty|\partial_x^\zeta u|_2\le\frac{\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2+Cc_0^2|\partial_x^\zeta u|_2^2,\\
J_{11}\le&C\int\Big(|\varphi\nabla^3 u||Lu||\nabla^3\varphi|+|\varphi\nabla Lu||\nabla^2\varphi||\nabla^3 u|\\
&\qquad+|\nabla\varphi|^2|\nabla Lu||\nabla^3 u|+|\varphi\nabla^2 Lu||\nabla\varphi||\nabla^3 u|\Big)\mathrm{d}x\\
\le&C\Big(|\nabla^3\varphi|_2|\nabla^2 u|_3|\varphi\nabla^3 u|_6+|\nabla^2\varphi|_3|\nabla^3 u|_2|\varphi\nabla^3 u|_6\\
&\qquad+|\nabla\varphi|_\infty^2|\nabla^3 u|_2^2+|\varphi\nabla^4 u|_2|\nabla\varphi|_\infty|\nabla^3 u|_2\Big)\\
\le&\frac{\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2+C\Big(|\nabla^3\varphi|_2^2|\nabla^2 u|_2|\nabla^3 u|_2+|\nabla^2\varphi|_3^2|\nabla^3 u|_2^2+|\nabla\varphi|_\infty^2|\nabla^3 u|_2^2\\
&\qquad+|\nabla^3\varphi|_2|\nabla\varphi|_\infty|\nabla^2 u|_2^{1/2}|\nabla^3 u|_2^{3/2}+|\nabla\varphi|_\infty|\nabla^2\varphi|_3||\nabla^3 u|_2^2\Big)\\
\le&\frac{\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2+C\big(c_0^2|\nabla^3 u|_2^2+c_0^4\big)\\
J_{12}\le&C\left(|\nabla\varphi|_\infty^2|\nabla^3 u|_2+|\varphi\nabla^3 u|_6|\nabla^2\varphi|_3+|\nabla\varphi|_\infty|\varphi\nabla^4 u|_2\right)|\nabla^3 v|_2\\
\le&\frac{\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2+C\Big((|\nabla^2\varphi|_3^2+|\nabla\varphi|_\infty^2)|\nabla^3 v|_2^2+|\nabla\varphi|_\infty^2|\nabla^3 v|_2|\nabla^3 u|_2\\
&\qquad+|\nabla\varphi|_\infty|\nabla^2\varphi|_3|\nabla^3 u|_2\Big)\\
\le&\frac{\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2+C(c_4^3|\nabla^3 u|_2+c_4^4).
\end{aligned}
$$

For J_{13}, we have

$$\begin{aligned}J_{13} \leq & C\int\Big(\left(|\nabla^2\varphi^2||\nabla^2 Q(v)|+|\nabla^3\varphi^2||\nabla Q(v)|\right)|\nabla^3 u|+\nabla\partial_x^\zeta\varphi^2\cdot Q(v)\partial_x^\zeta u\Big)\,\mathrm{d}x\\ \triangleq & J_{131}+J_{132}+J_{133}.\end{aligned}$$

with the natural correspondence for $J_{13i}(i=1,2,3)$. Using Hölder's inequality, Young's inequality, we first get

$$\begin{aligned}& J_{131}+J_{132}\\ \leq & C\Big(|\nabla^3\varphi|_2|\varphi\nabla^3 u|_6|\nabla^2 v|_3+|\nabla\varphi|_\infty|\nabla^2\varphi|_3|\nabla^3 u|_2|\nabla^2 v|_6\\ & \qquad +|\nabla^2\varphi|_3|\varphi\nabla^3 u|_6|\nabla^3 u|_2+|\nabla\varphi|_\infty^2|\nabla^3 v|_2|\nabla^3 u|_2\Big)\\ \leq & \frac{\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2+C\Big(|\nabla^3\varphi|_2^2|\nabla^2 v|_3^2+(|\nabla^2\varphi|_3^2+|\nabla\varphi|_\infty|\nabla^2\varphi|_3)|\nabla^3 u|_2^2\\ & +(|\nabla\varphi|_\infty^2|\nabla^3 v|_2+|\nabla\varphi|_\infty|\nabla^2\varphi|_3|\nabla^2 v|_6+|\nabla^3\varphi|_2|\nabla^2 v|_3|\nabla\varphi|_\infty)|\nabla^3 u|_2\Big)\\ \leq & \frac{\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2+C(c_4^3|\nabla^3 u|_2^2+c_4^4).\end{aligned}\tag{3.32}$$

Next we turn to estimating J_{133}. Denote $\zeta=\zeta^1+\zeta^2+\zeta^3$ with $\zeta^i\in\mathbb{R}^3(i=1,2,3)$ the three multi-indexes satisfying $|\zeta^i|=1$. after Integrating by parts, we obtain

$$\begin{aligned}J_{133}= & -\int\sum_{i=1}^3\partial_x^{\zeta-\zeta^i}\nabla\varphi^2\cdot\partial_x^{\zeta^i}Q(v)\partial_x^\zeta u\mathrm{d}x-\int\sum_{i=1}^3(\partial_x^{\zeta-\zeta^i}\nabla\varphi^2\cdot Q(v))\partial_x^{\zeta+\zeta^i}u\mathrm{d}x\\ \triangleq & \sum_{i=1}^3 J_{1331}^i+\sum_{i=1}^3 J_{1332}^i.\end{aligned}$$

with the natural correspondence for J_{1331}^i and J_{1332}^i $(i=1,2,3)$. We first consider the case $i=1$. It follows from the Hölder's inequality and Young's inequality that

$$\begin{aligned}J_{1331}^1= & -\int\partial_x^{\zeta^2+\zeta^3}\nabla\varphi^2\cdot\partial_x^{\zeta^1}Q(v)\cdot\partial_x^\zeta u\mathrm{d}x\\ \leq & C\left(|\nabla^3\varphi|_2|\nabla^2 v|_3|\varphi\nabla^3 u|_6+|\nabla\varphi|_\infty|\nabla^2\varphi|_6|\nabla^2 v|_3|\nabla^3 u|_2\right)\\ \leq & \frac{\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2+C(c_4^3|\nabla^3 u|_2+c_4^4),\\ J_{1332}^1= & -2\int\partial_x^{\zeta^2+\zeta^3}\nabla\varphi\varphi\cdot Q(v)\cdot\partial_x^{\zeta+\zeta^1}u\mathrm{d}x-2\int\partial_x^{\zeta^2}\nabla\varphi\partial_x^{\zeta^3}\varphi\cdot Q(v)\cdot\partial_x^{\zeta+\zeta^1}u\mathrm{d}x\\ & -2\int\partial_x^{\zeta^3}\nabla\varphi\partial_x^{\zeta^2}\varphi\cdot Q(v)\cdot\partial_x^{\zeta+\zeta^1}u\mathrm{d}x-2\int\nabla\varphi\partial_x^{\zeta^2+\zeta^3}\varphi\cdot Q(v)\cdot\partial_x^{\zeta+\zeta^1}u\mathrm{d}x\\ \triangleq & \sum_{i=1}^4 J_{1332}^{1i},\end{aligned}\tag{3.33}$$

with the natural correspondence for $J_{1332}^{1i} (i = 1, 2, 3, 4)$ which are estimated as follows.

$$J_{1332}^{11} \le C|\nabla^3\varphi|_2|\nabla v|_\infty|\varphi\nabla^4 u|_2 \le \frac{\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 + Cc_4^4,$$

$$\begin{aligned}
J_{1332}^{12} &= 2\int \partial_x^{\zeta^1}\left(\partial_x^{\zeta^2}\nabla\varphi\partial_x^{\zeta^3}\cdot Q(v)\right)\cdot\partial_x^{\zeta}u\mathrm{d}x \\
&\le C\int \left(|\nabla^3 u||\nabla v|(|\nabla^2\varphi|^2+|\nabla\varphi||\nabla^3\varphi|)+|\nabla^3 u||\nabla^2 v||\nabla^2\varphi||\nabla\varphi|\right)\mathrm{d}x \\
&\le C\left(|\nabla^2\varphi|_3|\nabla^2\varphi|_6|\nabla^3 u|_2|\nabla v|_\infty + |\nabla^3 u|_2|\nabla v|_\infty|\nabla\varphi|_\infty + |\nabla^3 u|_2|\nabla^2 v|_3|\nabla^2\varphi|_6|\nabla\varphi|_\infty\right) \\
&\le C\left(c_4^4|\nabla^3 u|_2^2 + c_4^2\right).
\end{aligned}$$

Similarly to J_{1332}^{12}, we have

$$J_{1332}^{13} + J_{1332}^{14} \le C\left(c_4^4|\nabla^3 u|_2^2 + c_4^2\right).$$

Then we obtain

$$J_{1332}^{1} \le \frac{\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 + C\left(c_4^4|\nabla^3 u|_2^2 + c_4^4\right).$$

Similarly we can get the same estimates for J_{1331}^{i} and J_{1332}^{i} $(i = 2, 3)$. With all these estimates, we arrive at

$$J_{133} \le \frac{3\alpha}{16}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 + C\left(c_4^4|\nabla^3 u|_2^2 + c_4^4\right). \tag{3.34}$$

Summing up (3.32) and (3.34), we obtain

$$J_{13} \le \frac{7\alpha}{32}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 + C\left(c_4^4|\nabla^3 u|_2^2 + c_4^4\right). \tag{3.35}$$

Now we are left with estimating J_{14}. It follows from (3.24) and Lemma 2.5 that

$$\begin{aligned}
J_{14} \le& C|\nabla^3 u|_2\Big(\|\nabla^3\overline{\sigma}_e\|_{L^1(\mathbb{R}^+\times S^2;L^2)} + \big(1+\|(\overline{\sigma}_a,\nabla\overline{\sigma}_a)\|_{L^2(\mathbb{R}^+\times S^2;L^\infty)}\big)\|\nabla^3 I\|_{L^2(\mathbb{R}^+\times S^2;L^2)} \\
&+ \|\nabla\overline{\sigma}_a\|_{L^2(\mathbb{R}^+\times S^2;L^\infty)}\|\nabla^2 I\|_{L^2(\mathbb{R}^+\times S^2;L^2)} + \|\nabla^3\overline{\sigma}_a\|_{L^2(\mathbb{R}^+\times S^2;L^2)}\|I\|_{L^2(\mathbb{R}^+\times S^2;L^\infty)}\Big) \\
\le& C\big(M^2(c_0)c_0^2|\nabla^3 u|_2^2 + c_0^2\big).
\end{aligned} \tag{3.36}$$

Substituting the estimates for $J_i (i = 8, \cdots, 14)$ into (3.31) and and summing up all $|\zeta| = 3$, we have

$$\frac{d}{dt}|\nabla^3 U|_2^2 + |\sqrt{(\varphi^2+\eta^2)}\nabla^4 u|_2^2 \le C\big(M^2(c_0)c_4^4|\nabla^3 u|_2^2 + c_4^4\big). \tag{3.37}$$

Applying Gronwall's inequality to (3.37), we arrive at

$$|\nabla^3 U|_2^2 + \int_0^t |\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2\mathrm{d}s \le C(c_0^2 + c_4^4 t)\exp(CM^2(c_0)c_4^4 t) \le Cc_0^2, \tag{3.38}$$

for $0 \le t \le T_3$.

Now it remains to prove the second estimate in (3.30) of Lemma 3.5. We already know, from (3.28) in the proof of Lemma 3.4, that

$$|u_t|_{D^1} \le C\left(M(c_0)c_3^2 + c_0|\sqrt{\varphi^2+\eta^2}\nabla^3 u|_2\right) \le CM(c_0)c_3^2. \tag{3.39}$$

For the D^2 norms, according to equations $(3.7)_3$ and $(3.7)_4$, one has

$$
\begin{aligned}
|\phi_t|_{D^2} \le & C|v\cdot\nabla\phi+\psi\operatorname{div}u|_{D^2}\le C(\|v\|_2\|\nabla\phi\|_2+\|\psi\|_2\|\nabla u\|_2)\le Cc_3^2,\\
|u_t|_{D^2}= & \left|-v\cdot\nabla u-\frac{\gamma-1}{2}\psi\nabla\phi-(\varphi^2+\eta^2)Lu+\nabla\varphi^2\cdot Q(v)-\frac{1}{c}\int_0^\infty\int_{S^2}\overline{A}_r\Omega\mathrm{d}\Omega\mathrm{d}\nu\right|_{D^2}\\
\le & C\Big(\|v\|_3\|\|\nabla u\|_2+|\nabla^2\psi|_2|\nabla\phi|_\infty+|\nabla\psi|_\infty|\nabla^2\phi|_2+|\psi|_\infty|\nabla^3\phi|_2\\
&+(|\varphi|_\infty+\|\nabla\varphi\|_2)(\|u\|_3+\|v\|_3)+|\sqrt{\varphi^2+\eta^2}|_\infty|\sqrt{\varphi^2+\eta^2}\nabla^4u|_2\\
&+\|\nabla^2\overline{\sigma}_e\|_{L^1(\mathbb{R}^+\times S^2;L^2)}+\left(1+\|(\nabla\overline{\sigma}_a,\nabla^2\overline{\sigma}_a)\|_{L^2(\mathbb{R}^+\times S^2;L^2)}\right)\|\nabla I\|_{L^2(\mathbb{R}^+\times S^2;H^1)}\Big)\\
\le & C\left(M(c_0)c_4^2+c_0|\sqrt{\varphi^2+\eta^2}\nabla^4u|_2\right),
\end{aligned}
\tag{3.40}
$$

which implies that

$$
\int_0^{T_4}|u_t|_{D^2}^2\mathrm{d}s\le C\int_0^{T_4}\left(M^2(c_0)c_4^4+c_0^2|\sqrt{\varphi^2+\eta^2}\nabla^4u|_2^2\right)\mathrm{d}s\le Cc_0^4. \tag{3.41}
$$

The proof of Lemma 3.5 is finished. □

From the above estimates in Lemmas 3.2-3.5, we know that

$$
\begin{gathered}
1+|\varphi|_\infty^2+\|\varphi\|_3^2\le Cc_0^2,\\
\|I\|_{L^2(\mathbb{R}^+\times S^2;H^3)}\le Cc_0,\quad \|I_t\|_{L^2(\mathbb{R}^+\times S^2;H^2)}\le CM(c_0)c_0^{l+1}c_1,\\
\|U(t)\|_2^2+\int_0^t|\sqrt{\varphi^2+\eta^2}\nabla^3u|_2^2\mathrm{d}s\le Cc_0^2,\\
|U|_{D^3}^2+\int_0^t|\sqrt{\varphi^2+\eta^2}\nabla^4u|_2^2\mathrm{d}s\le Cc_0^2,\\
\|U_t\|_1^2+\|\varphi_t\|_2^2+|\phi_t|_{D^2}^2+\int_0^t|u_t|_{D^1\cap D^2}^2\mathrm{d}s\le CM^2(c_0)c_4^6,
\end{gathered}
\tag{3.42}
$$

for $0\le t\le T_3=\min\{T_2,(1+M(c_0)c_0^lc_4)^{-4}\}$. Noticing that

$$
\begin{aligned}
T_3&=\min\{T^*,(1+c_4)^{-2},(1+M(c_0)c_0^lc_1)^{-2},(1+M(c_0)c_0^lc_4)^{-4}\}\\
&=\min\{T^*,(1+M(c_0)c_0^lc_4)^{-4}\},
\end{aligned}
$$

then we can define the constants $c_i(i=1,\cdots,5)$ and T^* as follows:

$$
\begin{aligned}
&c_1=C^{1/2}c_0,\ c_2=c_3=c_4=CM(c_0)c_0^{l+1}c_1=C^{\frac{3}{2}}M(c_0)c_0^{l+2},\\
&c_5=C^{\frac{1}{2}}M(c_0)c_4^3=C^5M^4(c_0)c_0^{3(l+2)},\\
&T^*=\min\{T,(1+M(c_0)c_0^lc_4)^{-4}\}.
\end{aligned}
\tag{3.43}
$$

Thus it turns out that

$$\begin{aligned}
\sup_{0\le t\le T^*}\|I\|_{L^2(\mathbb{R}^+\times S^2;H^3)}\le c_1,\ \sup_{0\le t\le T^*}\|I_t\|_{L^2(\mathbb{R}^+\times S^2;H^2)}\le c_2,\\
\sup_{0\le t\le T^*}\left(\|\varphi\|_2^2+\|U\|_2^2\right)+\int_0^{T^*}|\varphi\nabla^3 u|_2^2\mathrm{d}t\le c_3^2,\\
\sup_{0\le t\le T^*}\left(|\varphi|_{D^3}^2+|U|_{D^3}^2\right)+\int_0^{T^*}|\varphi\nabla^4 u|_2^2\mathrm{d}t\le c_4^2,\\
\sup_{0\le t\le T^*}\left(\|U_t\|_1^2+\|\varphi_t\|_2^2+|\phi_t|_{D^2}^2\right)+\int_0^{T^*}|u_t|_{D^1\cap D^2}^2\mathrm{d}s\le c_5^2,
\end{aligned}\tag{3.44}$$

for $0\le t\le T^*$.

3.2. **Passing to the limit as $\eta\to 0$.** The purpose of this section is to obtain the local existence of the following degenerate linearized problem without an artificial viscosity when $\varphi_0,\phi_0\ge 0$,

$$\begin{cases}
\dfrac{1}{c}I_t+\Omega\cdot\nabla I=\widetilde{A}_r,\\
\varphi_t+v\cdot\nabla\varphi+\dfrac{\delta-1}{2}\omega\operatorname{div}v=0,\\
U_t+\displaystyle\sum_{j=1}^3 A_j(V)U_{x_j}=-\varphi^2F(U)+G(\varphi,V)+H(I,\varphi),\\
(I,\varphi,U)|_{t=0}=(I_0,\varphi_0,U_0),\\
(I,\varphi,U)\to(0,0,0),\quad \text{as}\quad |x|\to+\infty,\ t\ge 0.
\end{cases}\tag{3.45}$$

Theorem 3.2. *Assume that the initial data satisfy* (3.5). *Then there exist a time $T^*>0$ and a unique strong solution (I,φ,U) to* (3.45) *satisfying* (3.9) *in $(t,x,\nu,\Omega)\in(0,T^*]\times\mathbb{R}^3\times\mathbb{R}^+\times S^2$. Moreover, (I,φ,U) also satisfies the estimates in* (3.44).

Proof. From Lemma 3.1 and (3.43)-(3.44), we know that for every fixed $\eta>0$, there exist a time T^* independent of η and a unique strong solution $(I^\eta,\varphi^\eta,U^\eta)$ to the linearized problem (3.7) in $(0,T^*]\times\mathbb{R}^3\times\mathbb{R}^+\times S^2$ satisfying the uniform a *priori* estimates in (3.44). By using the Aubin-Lions Lemma, one can obtain the corresponding compactness of solutions. For any $R>0$, there exists a subsequence of solutions (still denoted by $(I^\eta,\varphi^\eta,U^\eta)$ for simplicity), which converges to a limit (I,φ,U) in the following sense:

$$\begin{aligned}
I^\eta\to I\quad\text{in}\ L^2(\mathbb{R}^+\times S^2;C([0,T^*];H^2(B_R))),\quad\text{as}\quad\eta\to 0,\\
(\varphi^\eta,U^\eta)\to(\varphi,U)\quad\text{in}\ C([0,T^*];H^2(B_R)),\quad\text{as}\quad\eta\to 0,
\end{aligned}\tag{3.46}$$

where B_R is a ball centered at the origin with radius R. Furthermore, based on the uniform estimates in (3.44), we also know that there exists a subsequence of solutions (still denoted by $(I^\eta,\varphi^\eta,U^\eta)$ for simplicity), which converges to (I,φ,U) in the following sense:

$$\begin{aligned}
I^\eta &\overset{*}{\rightharpoonup} I \quad \text{weakly* in } L^2(\mathbb{R}^+\times S^2; L^\infty([0,T^*];H^3)),\\
I_t^\eta &\overset{*}{\rightharpoonup} I_t \quad \text{weakly* in } L^2(\mathbb{R}^+\times S^2; L^\infty([0,T^*];H^2)),\\
(\varphi^\eta, U^\eta) &\overset{*}{\rightharpoonup} (\varphi, U) \quad \text{weakly* in } L^\infty([0,T^*];H^3),\\
(\varphi_t^\eta, \phi_t^\eta) &\overset{*}{\rightharpoonup} (\varphi_t, \phi_t) \quad \text{weakly* in } L^\infty([0,T^*];H^2),\\
u_t^\eta &\overset{*}{\rightharpoonup} u_t \quad \text{weakly* in } L^\infty([0,T^*];H^1),\\
u_t^\eta &\rightharpoonup u_t \quad \text{weakly in } L^2([0,T^*];D^2).
\end{aligned} \tag{3.47}$$

Combining (3.47) and (3.46), we have

$$\varphi^\eta\nabla^4 u^\eta \rightharpoonup \varphi\nabla^4 u \quad \text{weakly in } L^2([0,T^*];L^2). \tag{3.48}$$

Using (3.46)-(3.48), one can easily show that (I,φ,U) is a weak solution in the sense of distribution to the linearized problem (3.45), and satisfies

$$\begin{aligned}
&I\in L^2(\mathbb{R}^+\times S^2; L^\infty([0,T^*];H^3)), \quad I_t\in L^2(\mathbb{R}^+\times S^2; L^\infty([0,T^*];H^2)),\\
&(\varphi,\phi)\in L^\infty([0,T^*];H^3), \quad (\varphi_t,\phi_t)\in L^\infty([0,T^*];H^2),\\
&u\in L^\infty([0,T^*];H^{s'})\cap L^\infty([0,T^*];H^3), \quad \varphi\nabla^4 u\in L^2([0,T^*];L^2)\\
&u_t\in L^\infty([0,T^*];H^1)\cap L^2([0,T^*];D^2),
\end{aligned} \tag{3.49}$$

for any constant $s'\in[2,3)$.

In order to show that the weak solution (I,φ,U) is also a strong solution, we need to prove the time continuity of solutions. We first prove the time continuity of φ. Using the classical Sobolev embedding, we have

$$\varphi\in C([0,T^*];H^2)\cap C([0,T^*];H^3\text{-weak}).$$

According to the proof of Lemma 3.2, we get

$$\limsup_{t\to 0}\|\varphi\|_3\le\|\varphi_0\|_3,$$

which implies that φ is right continuous at $t=0$ in H^3. Therefore, one can obtain $\varphi\in C([0,T^*];H^3)$ by using the time reversibility of the equation $(3.45)_2$.

Using the fact that $\omega\nabla v\in L^2([0,T^*];H^3)$ and $(\omega\nabla v)_t\in L^2([0,T^*];H^1)$, we derive $\omega\nabla v\in C([0,T^*];H^2)$. This, together with $(3.45)_2$, leads to $\varphi_t\in C([0,T^*];H^2)$. The time continuity of ϕ and I can be obtained similarly.

For the time continuity of u, from (3.44) one has

$$u\in C([0,T^*];H^2)\cap C([0,T^*];H^3\text{-weak}).$$

It follows from Lemma 2.2 that for any constant $s'\in[2,3)$,

$$\|u\|_{s'}\le C\|u\|_0^{1-\frac{s'}{s}}\|u\|_s^{\frac{s'}{s}},$$

which, along with the uniform estimates in (3.44), yields $u\in C([0,T^*];H^{s'})$. It also follows from (3.44) that

$$\varphi^2 Lu\in L^2([0,T^*];H^2),\ (\varphi^2 Lu)_t\in L^2([0,T^*];L^2),$$

which, together with the Aubin-Lions lemma, implies that $\varphi^2 Lu \in C([0,T^*];H^1)$. Then we obtain from $(3.45)_4$ that $u_t \in C([0,T^*];H^1)$.

For the uniqueness of solutions, one can use the same arguments as in the proof of Lemma 3.1, we omit the details here. The proof of Theorem 3.2 is finished. □

3.3. **Local well-posedness of nonlinear problem: proof of Theorem 3.1.** In this section, based on the previous linearized result, we will use the classical iteration scheme to establish the local-in-time existence. We first assume that

$$2+|\varphi_0|_\infty+\|I_0\|_{L^2(\mathbb{R}^+\times S^2;H^3)}+\|(\varphi_0,U_0)\|_3\le c_0.$$

Let $(I^0,\varphi^0,\phi^0,u^0)$ be the solution to the following problem in $(0,+\infty)\times\mathbb{R}^3\times\mathbb{R}^+\times S^2$:

$$\begin{cases}\frac{1}{c}W_t+\Omega\cdot\nabla W=0,\\ X_t+u_0\cdot\nabla X=0,\\ Y_t+u_0\cdot\nabla Y=0,\\ Z_t-X^2\triangle Z=0,\\ (W,X,Y,Z)|_{t=0}=(I_0,\varphi_0,\phi_0,u_0),\\ (W,X,Y,Z)\to(0,0,0,0),\quad \text{as}\quad |x|\to+\infty,\quad t>0.\end{cases}\tag{3.50}$$

Then, we can choose $T^{**}\in(0,T^*]$ and constants $c_i(i=1,\cdots,5)$ such that

$$\begin{aligned}\sup_{0\le t\le T^{**}}\|I^0\|_{L^2(\mathbb{R}^+\times S^2;H^3)}\le c_1,\ \sup_{0\le t\le T^{**}}\|I^0_t\|_{L^2(\mathbb{R}^+\times S^2;H^2)}\le c_2,\\ \sup_{0\le t\le T^{**}}\left(\|(\varphi^0,\phi^0)\|_2^2+\|u^0\|_2^2\right)+\int_0^{T^{**}}|\varphi^0\nabla^3u^0|_2^2\mathrm{d}t\le c_3^2,\\ \sup_{0\le t\le T^{**}}\left(|(\varphi^0,\phi^0)|_{D^3}^2+|u^0|_{D^3}^2\right)+\int_0^{T^{**}}|\varphi^0\nabla^4u^0|_2^2\mathrm{d}t\le c_4^2,\\ \sup_{0\le t\le T^{**}}\left(\|(\phi^0_t,u^0_t)\|_1^2+\|\varphi^0_t\|_2^2+|\phi^0_t|_{D^2}^2\right)+\int_0^{T^{**}}|u^0_t|_{D^1\cap D^2}^2\mathrm{d}s\le c_5^2.\end{aligned}\tag{3.51}$$

We now give the detailed proof of Theorem 3.1.

Proof. Denote $U^k=(\phi^k,u^k)(k=0,1,\cdots)$. If we assume in (3.45) that $(\chi,\omega,V)=(I^0,\varphi^0,U^0)$, then we can obtain a strong solution (I^1,φ^1,U^1) of (3.45). For any given (I^k,φ^k,U^k), we can construct the approximate sequence of solutions $(I^{k+1},\varphi^{k+1},U^{k+1})$

by solving the following problem:

$$\begin{cases} \frac{1}{c} I_t^{k+1} + \Omega \cdot \nabla I^{k+1} = A_r^k, \\ \varphi_t^{k+1} + u^k \cdot \nabla \varphi^{k+1} + \frac{\delta - 1}{2} \varphi^k \operatorname{div} u^k = 0, \\ U_t^{k+1} + \sum_{j=1}^{3} A_j(U^k) U_{x_j}^{k+1} = -(\varphi^{k+1})^2 F(U^{k+1}) + G(\varphi^{k+1}, U^k) + H(I^{k+1}, \varphi^{k+1}), \\ (I^{k+1}, \varphi^{k+1}, U^{k+1})|_{t=0} = (I_0, \varphi_0, U_0), \\ (I^{k+1}, \varphi^{k+1}, U^{k+1}) \to (0,0,0), \quad \text{as} \quad |x| \to +\infty, \quad t \geq 0, \end{cases} \tag{3.52}$$

where

$$\begin{aligned} & A_r^k = \sigma_e^{k+1} - \sigma_a^{k+1} I^{k+1} + \int_0^\infty \int_{S^2} \left(\frac{\nu}{\nu'} \sigma_s^{k+1} I'^k - (\sigma_s')^{k+1} I^{k+1} \right) \mathrm{d}\Omega' \mathrm{d}\nu', \\ & H(I^{k+1}, \varphi^{k+1}) = \left(0, -\frac{1}{c} \int_0^\infty \int_{S^2} \overline{A}_r^k \Omega \mathrm{d}\Omega \mathrm{d}\nu \right)^\top, \\ & \overline{A}_r^k = \overline{\sigma}_e^{k+1} - \overline{\sigma}_a^{k+1} I^{k+1} + \int_0^\infty \int_{S^2} \left(\frac{\nu}{\nu'} \overline{\sigma}_s I'^{k+1} - \overline{\sigma}_s' I^{k+1} \right) \mathrm{d}\Omega' \mathrm{d}\nu', \\ & \sigma_e^{k+1} = \sigma_e(t, x, \nu, \Omega, (\varphi^{k+1})^l), \quad \sigma_a^{k+1} = \sigma_a(t, x, \nu, \Omega, (\varphi^{k+1})^l), \\ & \overline{\sigma}_e^{k+1} = \overline{\sigma}_e(t, x, \nu, \Omega, (\varphi^{k+1})^l), \quad \overline{\sigma}_a^{k+1} = \overline{\sigma}_a(t, x, \nu, \Omega, (\varphi^{k+1})^l), \\ & \sigma_s^{k+1} = \overline{\sigma}_s (\varphi^{k+1})^l, \quad (\sigma_s')^{k+1} = \overline{\sigma}_s' (\varphi^{k+1})^l, \quad \sigma_a^{k+1} = \overline{\sigma}_a (\varphi^{k+1})^l. \end{aligned}$$

It is not difficult to see that the sequence of solutions (I^k, φ^k, U^k) satisfy the uniform *a priori* estimates in (3.44) for $0 < t \leq T^{**}$. Next, we want to obtain the strong convergence of the approximate solution sequence (I^k, φ^k, U^k). Let

$$\begin{aligned} & \overline{I}^{k+1} = I^{k+1} - I^k, \ \overline{\varphi}^{k+1} = \varphi^{k+1} - \varphi^k, \\ & \overline{U}^{k+1} = (\overline{\phi}^{k+1}, \overline{u}^{k+1}), \text{ with } \overline{\phi}^{k+1} = \phi^{k+1} - \phi^k, \ \overline{u}^{k+1} = u^{k+1} - u^k. \end{aligned}$$

We obtain from (3.52) that

$$\begin{cases} \frac{1}{c} \overline{I}_t^{k+1} + \Omega \cdot \nabla \overline{I}^{k+1} + \left(\sigma_a^{k+1} + \int_0^\infty \int_{S^2} (\sigma_s')^{k+1} \mathrm{d}\Omega' \mathrm{d}\nu' \right) \overline{I}^{k+1} = L_1, \\ \overline{\varphi}_t^{k+1} + u^k \cdot \nabla \overline{\varphi}^{k+1} + \overline{u}^k \cdot \nabla \varphi^k + \frac{\delta - 1}{2} (\overline{\varphi}^k \operatorname{div} u^{k-1} + \varphi^k \operatorname{div} \overline{u}^k) = 0, \\ \overline{U}_t^{k+1} + \sum_{j=1}^{3} A_j(U^k) \overline{U}_{x_j}^{k+1} + (\varphi^{k+1})^2 F(\overline{U}^{k+1}) \\ \quad = -\sum_{j=1}^{3} A_j(\overline{U}^k) U_{x_j}^k - \overline{\varphi}^{k+1} (\varphi^{k+1} + \varphi^k) F(U^k) \\ \qquad + G(\overline{\varphi}^{k+1}, U^k) + G(\varphi^k, \overline{U}^k) + H(\overline{I}^{k+1}, \overline{\varphi}^{k+1}), \end{cases} \tag{3.53}$$

where L_1 is given by

$$
\begin{aligned}
L_1 =&(\sigma_e^{k+1}-\sigma_e^k)-I^k(\sigma_a^{k+1}-\sigma_a^k)-\int_0^\infty\int_{S^2}\left((\sigma_s')^{k+1}-(\sigma_s')^k\right)I^k\mathrm{d}\Omega'\mathrm{d}\nu'\\
&+\int_0^\infty\int_{S^2}\left(\frac{\nu}{\nu'}(\sigma_s^k\overline{I}'^k+I'^k(\sigma_s^{k+1}-\sigma_s^k))\right)\mathrm{d}\Omega'\mathrm{d}\nu',
\end{aligned}
$$

and $H(\overline{I}^{k+1},\overline{\varphi}^{k+1})=(0,L_2)^\top$, with

$$
\begin{aligned}
L_2 =&-\frac{1}{c}\int_0^\infty\int_{S^2}\left((\overline{\sigma}_e^{k+1}-\overline{\sigma}_e^k)-\left(\overline{\sigma}_a^{k+1}+\int_0^\infty\int_{S^2}\overline{\sigma}_s'\right)\overline{I}^{k+1}\mathrm{d}\Omega'\mathrm{d}\nu'\right)\Omega\mathrm{d}\Omega\mathrm{d}\nu\\
&-\frac{1}{c}\int_0^\infty\int_{S^2}I^k(\overline{\sigma}_a^{k+1}-\overline{\sigma}_a^k)\Omega\mathrm{d}\Omega\mathrm{d}\nu\\
&-\frac{1}{c}\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\frac{\nu}{\nu'}\overline{\sigma}_s(I')^{k+1}\Omega\mathrm{d}\Omega'\mathrm{d}\nu'\mathrm{d}\Omega\mathrm{d}\nu.
\end{aligned}
$$

First, we estimate $|\overline{\varphi}^{k+1}|_2$. Multiplying $(3.53)_2$ by $2\overline{\varphi}^{k+1}$, integrating over $\mathbb{R}^3$, and using Young's inequality, one has

$$
\begin{aligned}
\frac{d}{dt}|\overline{\varphi}^{k+1}|_2^2\le& C\left(|\nabla u^k|_\infty|\overline{\varphi}^{k+1}|_2^2+\left(|\nabla\varphi^k|_\infty|\overline{u}^k|_2+|\varphi^k\nabla\overline{u}^k|_2+|\overline{\varphi}^k|_2|\nabla u^{k-1}|_\infty\right)|\overline{\varphi}^{k+1}|_2\right)\\
\le& A_\epsilon^k(t)|\overline{\varphi}^{k+1}|_2^2+\epsilon\left(|\overline{u}^k|_2^2+|\varphi^k\nabla\overline{u}^k|_2^2+|\overline{\varphi}^k|_2^2\right),
\end{aligned}
\tag{3.54}
$$

for some $A_\epsilon^k(t)$ satisfying $\int_0^t A_\epsilon^k(s)\mathrm{d}s\le C_\epsilon t$, and C_ϵ is a positive constant depending on ϵ.

Now we estimate $|\overline{I}^{k+1}|_{L^2(\mathbb{R}^+\times S^2;L^2)}$. Multiplying $(3.53)_1$ by $c\overline{I}^{k+1}$ and integrating over $\mathbb{R}^+\times S^2\times\mathbb{R}^3$, we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\|\overline{I}^{k+1}\|_{L^2(\mathbb{R}^+\times S^2;L^2)}^2\\
\le& C\Bigg(\int_0^\infty\int_{S^2}\left(|\sigma_e^{k+1}-\sigma_e^k|_2|\overline{I}^{k+1}|_2+|I^k|_\infty|\sigma_a^{k+1}-\sigma_a^k|_2|\overline{I}^{k+1}|_2\right)\mathrm{d}\Omega\mathrm{d}\nu\\
&+\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\Big(\frac{\nu}{\nu'}\overline{\sigma}_s\big(|(\varphi^k)^l|_\infty|\overline{I}'^k|_2+|I'^k|_\infty|(\varphi^{k+1})^l-(\varphi^k)^l|_2\big)|\overline{I}'^{k+1}|_2\\
&\qquad+\overline{\sigma}_s'|I^k|_\infty|(\varphi^{k+1})^l-(\varphi^k)^l|_2|\overline{I}^{k+1}|_2\Big)\mathrm{d}\Omega'\mathrm{d}\nu'\mathrm{d}\Omega\mathrm{d}\nu\Bigg)\\
\le& C|\overline{\varphi}^{k+1}|_2^2+\epsilon\|\overline{I}^k\|_{L^2(\mathbb{R}^+\times S^2;L^2)}^2+C_\epsilon\|\overline{I}^{k+1}\|_{L^2(\mathbb{R}^+\times S^2;L^2)}^2,
\end{aligned}
\tag{3.55}
$$

where we used the Hypothesis **H1** and (2.6)-(2.7).

Now we turn to the estimate of $|\overline{U}^{k+1}|_2$. Multiplying $(3.53)_3$ by $2\overline{U}^{k+1}$ and integrating over $\mathbb{R}^3$, we have

$$
\begin{aligned}
&\frac{d}{dt}|\overline{U}^{k+1}|_2^2+2\alpha|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2+2(\alpha+\beta)|\varphi^{k+1}\operatorname{div}\overline{u}^{k+1}|_2^2\\
=&\int(\overline{U}^{k+1})^\top\operatorname{div}A(U^k)\overline{U}^{k+1}\mathrm{d}x-2\sum_{j=1}^{3}\int(\overline{U}^{k+1})^\top A_j(\overline{U}^k)\partial_jU^k\mathrm{d}x\\
&-2\int\nabla(\varphi^{k+1})^2\cdot\Big(\alpha\nabla\overline{u}^{k+1}+(\alpha+\beta)\operatorname{div}\overline{u}^{k+1}\mathbb{I}_3\Big)\cdot\overline{u}^{k+1}\mathrm{d}x\\
&-2\int\overline{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k)\cdot Lu^k\cdot\overline{u}^{k+1}\mathrm{d}x+2\int\nabla(\overline{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k))\cdot Q(u^k)\cdot\overline{u}^{k+1}\mathrm{d}x\\
&+2\int\nabla(\varphi^k)^2\cdot(Q(u^k)-Q(u^{k-1}))\cdot\overline{u}^{k+1}\mathrm{d}x+2\int H(\overline{I}^{k+1},\overline{\varphi}^{k+1})\cdot\overline{U}^{k+1}\mathrm{d}x\\
\triangleq&\sum_{i=15}^{21}J_i,
\end{aligned}
\tag{3.56}
$$

with the natural correspondence for $J_i(i=15,16,\cdots,21)$. Now we first estimate $J_i(i=15,16,\cdots,20)$ one by one. It follows from Gagliardo-Nirenberg inequality, Hölder's inequality and Young's inequality that

$$
\begin{aligned}
J_{15}\le&C|\nabla U^k|_\infty|\overline{U}^{k+1}|_2^2\le C|\overline{U}^{k+1}|_2^2,\\
J_{16}\le&C|\nabla U^k|_\infty|\overline{U}^{k+1}|_2|\overline{U}^k|_2\le C_\epsilon|\overline{U}^{k+1}|_2^2+\epsilon|\overline{U}^k|_2^2,\\
J_{17}\le&C\int|\varphi^{k+1}\nabla\overline{u}^{k+1}||\nabla\varphi^{k+1}||\overline{u}^{k+1}|\mathrm{d}x\le\frac{\alpha}{32}|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2+C|\overline{u}^{k+1}|_2^2|\nabla\varphi^{k+1}|_\infty^2,\\
J_{18}\le&C|\overline{\varphi}^{k+1}|_2\Big(|\varphi^{k+1}\overline{u}^{k+1}|_6|Lu^k|_3+|\varphi^kLu^k|_\infty|\overline{u}^{k+1}|_2\Big)\\
\le&C|\overline{\varphi}^{k+1}|_2\Big(|\nabla(\varphi^{k+1}\overline{u}^{k+1})|_2|Lu^k|_3+\|\nabla(\varphi^k\nabla^2u^k)\|_1|\overline{u}^{k+1}|_2\Big)\\
\le&|\overline{\varphi}^{k+1}|_2\Big((|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2+|\nabla\varphi^{k+1}|_\infty|\overline{u}^{k+1}|_2)|\nabla^2u^k|_3\\
&+(\|\nabla\varphi^k\|_2\|\nabla^2u^k\|_1+|\varphi^k\nabla^4u^k|_2)|\overline{u}^{k+1}|_2\Big)\\
\le&\frac{\alpha}{32}|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2+C\Big(|\overline{\varphi}^{k+1}|_2^2(1+\|\nabla^2u^k\|_1^2)\\
&+|\overline{u}^{k+1}|_2^2(|\nabla\varphi^{k+1}|_\infty^2+\|\nabla\varphi^k\|_2^2\|\nabla^2u^k\|_1^2+|\varphi^k\nabla^4u^k|_2^2)\Big)\\
J_{19}\le&C\int|\overline{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k)|\Big(|\nabla^2u^k||\overline{u}^{k+1}|+|\nabla u^k||\nabla\overline{u}^{k+1}|\Big)\mathrm{d}x\\
\le&C\Bigg(|\overline{\varphi}^{k+1}|_2(|\nabla^2u^k|_3|\varphi^{k+1}\overline{u}^{k+1}|_6+|\varphi^k\nabla^2u^k|_\infty|\overline{u}^{k+1}|_2+|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2|\nabla u^k|_\infty)
\end{aligned}
$$

$$
\begin{aligned}
&+\int|\overline{\varphi}^{k+1}||\varphi^k-\varphi^{k+1}+\varphi^{k+1}||\nabla u^k||\nabla\overline{u}^{k+1}|\mathrm{d}x\Bigg)\\
\leq& C|\overline{\varphi}^{k+1}|_2\Bigg(|\nabla^2u^k|_3\left(|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2+|\nabla\varphi^k|_\infty|\overline{u}^{k+1}|_2\right)\\
&+|\overline{u}^{k+1}|_2\left(\|\nabla\varphi^k\|_2\|\nabla^2u^k\|_1+|\varphi^k\nabla^4u^k|_2\right)\\
&+|\nabla u^k|_\infty\left(|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2+|\overline{\varphi}^{k+1}|_2|\nabla\overline{u}^{k+1}|_\infty\right)\Bigg)\\
\leq&\frac{\alpha}{32}|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2+C\Bigg(|\overline{\varphi}^{k+1}|_2^2\left(1+|\nabla u^k|_\infty^2+|\nabla\overline{u}^{k+1}|_\infty^2+\|\nabla^2u^k\|_1^2\right)\\
&+|\overline{u}^{k+1}|_2^2\left(|\nabla\varphi^k|_\infty^2+\|\nabla\varphi^k\|_2^2\|\nabla^2u^k\|_1^2+|\varphi^k\nabla^4u^k|_2^2\right)\Bigg),\\
J_{20}\leq&\epsilon|\varphi^k\nabla\overline{u}^k|_2^2+C_\epsilon|\nabla\varphi^k|_\infty^2|\overline{u}^{k+1}|_2^2.
\end{aligned}
$$

We are left with the estimating of the last term J_{21} in (3.56). Recalling the definition of L_2 and using the Hypothesis **H1** and **H2**, we have

$$
\begin{aligned}
J_{21}=&2\int L_2\cdot\overline{u}^{k+1}\mathrm{d}x\\
=&-\frac{2}{c}\int\int_0^\infty\int_{S^2}\left(-\left(\overline{\sigma}_a^{k+1}+\int_0^\infty\int_{S^2}\overline{\sigma}_s'\right)\overline{I}^{k+1}\mathrm{d}\Omega'\mathrm{d}\nu'\right)\Omega\cdot\overline{u}^{k+1}\mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x\\
&-\frac{2}{c}\int\int_0^\infty\int_{S^2}\left((\overline{\sigma}_e^{k+1}-\overline{\sigma}_e^k)+I^k(\overline{\sigma}_a^{k+1}-\overline{\sigma}_a^k)\right)\Omega\cdot\overline{u}^{k+1}\mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x\\
&-\frac{2}{c}\int\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\frac{\nu}{\nu'}\overline{\sigma}_sI'^{k+1}\Omega\cdot\overline{u}^{k+1}\mathrm{d}\Omega'\mathrm{d}\nu'\mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x\\
\leq&C|\overline{u}^{k+1}|_2\Big(\|\overline{\sigma}_e^{k+1}-\overline{\sigma}_e^k\|_{L^1(\mathbb{R}^+\times S^2;L^2)}+(1+\|\overline{\sigma}_a^{k+1}\|_{L^2(\mathbb{R}^+\times S^2;H^2)})\|\overline{I}^{k+1}\|_{L^2(\mathbb{R}^+\times S^2;L^2)}\\
&+\|\overline{\sigma}_a^{k+1}-\overline{\sigma}_a^k\|_{L^2(\mathbb{R}^+\times S^2;L^2)}\|I^k\|_{L^2(\mathbb{R}^+\times S^2;H^2)}\Big)\\
\leq&C\left(|\overline{\varphi}^{k+1}|_2^2+|\overline{u}^{k+1}|_2^2+\|\overline{I}^{k+1}\|_{L^2(\mathbb{R}^+\times S^2;L^2)}^2\right),
\end{aligned}
$$

Substituting $J_i(i=15,\cdots,21)$ into (3.56) , we conclude that

$$
\begin{aligned}
&\frac{d}{dt}|\overline{U}^{k+1}|_2^2+\alpha|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2\\
\leq& B_\epsilon^k(t)|\overline{U}^{k+1}|_2^2+C\left(|\overline{\varphi}^{k+1}|_2^2+\|\overline{I}^{k+1}\|_{L^2(\mathbb{R}^+\times S^2;L^2)}^2\right)+\epsilon\left(|\overline{U}^k|_2^2+|\overline{\varphi}^k|_2^2+|\varphi^k\nabla\overline{u}^k|_2^2\right),
\end{aligned}
\tag{3.57}
$$

for some $B_\epsilon^k(t)$ satisfying $\int_0^t B_\epsilon^k(s)\mathrm{d}s\leq C+C_\epsilon t$.

From (3.54)-(3.55) and (3.57), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|\overline{I}^{k+1}\|^2_{L^2(\mathbb{R}^+\times S^2;L^2)}+|\overline{\varphi}^{k+1}|_2^2+|\overline{U}^{k+1}|_2^2\right)+\alpha|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2\\
\le&E^k_\epsilon(t)\left(|\overline{\varphi}^{k+1}|_2^2+|\overline{U}^{k+1}|_2^2+\|\overline{I}^{k+1}\|^2_{L^2(\mathbb{R}^+\times S^2;L^2)}\right)\\
&\quad+C\epsilon\left(|\varphi^k\nabla\overline{u}^k|_2+|\overline{\varphi}^k|_2^2+|\overline{U}^k|_2^2+\|\overline{I}^k\|^2_{L^2(\mathbb{R}^+\times S^2;L^2)}\right),
\end{aligned}\tag{3.58}$$

for some $E^k_\epsilon(t)$ satisfying $\int_0^t E^k_\epsilon(s)\mathrm{d}s\le C+C_\epsilon t$. Denote

$$\Gamma^{k+1}(t)=\sup_{s\in[0,t]}\|\overline{I}^{k+1}\|^2_{L^2(\mathbb{R}^+\times S^2;L^2)}+\sup_{s\in[0,t]}|\overline{\varphi}^{k+1}|_2^2+\sup_{s\in[0,t]}|\overline{U}^{k+1}|_2^2.$$

It thus follows from (3.58) that

$$\begin{aligned}
&\Gamma^{k+1}(t)+\alpha\int_0^t|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2\mathrm{d}s\\
\le&C\epsilon\int_0^t\left(|\varphi^k\nabla\overline{u}^k|_2^2+\|\overline{I}^k\|^2_{L^2(\mathbb{R}^+\times S^2;L^2)}+|\overline{\varphi}^k|_2^2+|\overline{U}^k|_2^2\right)\mathrm{d}s\exp(C_\epsilon t)\\
\le&C\epsilon\left(\int_0^t|\varphi^k\nabla\overline{u}^k|_2^2\mathrm{d}s+t\sup_{s\in[0,t]}\left(\|\overline{I}^k\|^2_{L^2(\mathbb{R}^+\times S^2;L^2)}+|\overline{\varphi}^k|_2^2+|\overline{U}^k|_2^2\right)\exp(C_\epsilon t)\right).
\end{aligned}\tag{3.59}$$

Then by choosing $\epsilon>0$ and $0<T_*<T^{**}$ suitably small such that

$$4C\epsilon\le\min(\alpha,1),\quad(1+T_*)\exp(C_\epsilon T_*)\le 2,$$

we finally arrive at

$$\sum_{k=1}^{\infty}\left(\Gamma^{k+1}(T_*)+\alpha\int_0^{T_*}|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2\mathrm{d}t\right)\le C<+\infty,\tag{3.60}$$

which, together with the uniform estimates in (3.44), yields

$$\begin{aligned}
&I^k\to I\quad\text{in } L^2(\mathbb{R}^+\times S^2;L^\infty([0,T_*];H^2)),\\
&(\varphi^k,U^k)\to(\varphi,U)\quad\text{in } L^\infty([0,T_*];H^2).
\end{aligned}\tag{3.61}$$

By virtue of the uniform estimates in (3.44), we also know that the approximate solution sequence (I^k,φ^k,U^k) has the same compactness as in (3.47), with the limit (I,φ,U) a weak solution to (3.2) in the sense of distribution and satisfying regularities

$$\begin{aligned}
&I\in L^2(\mathbb{R}^+\times S^2;L^\infty([0,T_*];H^3)),\quad I_t\in L^2(\mathbb{R}^+\times S^2;L^\infty([0,T_*];H^2)),\\
&(\varphi,\phi)\in L^\infty([0,T_*];H^3),\quad(\varphi_t,\phi_t)\in L^\infty([0,T_*];H^2),\\
&u\in L^\infty([0,T_*];H^{s'})\cap L^\infty([0,T_*];H^3),\quad\varphi\nabla^4u\in L^2([0,T_*];L^2),\\
&u_t\in L^\infty([0,T_*];H^1)\cap L^2([0,T_*];D^2),
\end{aligned}\tag{3.62}$$

for any constant $s'\in[2,3)$.

The time continuity and uniqueness of solutions can be obtained by using similar arguments as in Section 3.2. We omit the details here. □

4. PROOF OF THEOREM 2.1

With Theorem 3.1 at hand, we are ready to give the proof of Theorem 2.1. The proof will be divided into two steps.

Step 1. The existence of regular solutions to (1.8)-(1.9).

Step 2. The regular solution that we obtained is indeed a classical one for $t \in (0, T_*]$.

From Theorem 3.1 we know that there exists a unique strong solution (I, φ, ϕ, u) to (3.2) satisfying (3.6), which, along with the Sobolev embedding implies that

$$\begin{aligned} &(\varphi, \phi) \in C^1([0, T_*] \times \mathbb{R}^3), \quad (u, \nabla u) \in C((0, T_*] \times \mathbb{R}^3), \\ &(I, I_t) \in L^2(\mathbb{R}^+ \times S^2; C([0, T_*] \times \mathbb{R}^3)). \end{aligned} \tag{4.1}$$

Since $\rho = \varphi^{\frac{2}{\delta-1}}$ and $1 < \delta \leq \min\{\gamma, 5/3\}$, it is easy to get

$$\rho(t, x) \in C^1([0, T_*] \times \mathbb{R}^3), \tag{4.2}$$

and

$$\frac{\partial \rho}{\partial \varphi}(t, x) = \frac{2}{\delta - 1} \varphi^{\frac{3-\delta}{\delta-1}}(t, x) \in C([0, T_*] \times \mathbb{R}^3).$$

Then, after multiplying $(3.1)_2$ by $\frac{\partial \rho}{\partial \varphi}$, one can directly derive the continuity equation. The momentum equation $(1.8)_3$ can be obtained by multiplying $(3.1)_4$ by ρ. According to the continuity equation, we also know that $\rho(t, x) \geq 0$ provided the initial density $\rho_0 \geq 0$. By taking the limit as $\rho \to 0$ on both side of $(3.1)_4$, we can find that the velocity u can be governed by the following nonlinear equation:

$$u_t + u \cdot \nabla u = -\frac{1}{c} \int_0^\infty \int_{S^2} \left(\lim_{\rho \to 0} \overline{A}_r \right) \Omega \mathrm{d}\Omega \mathrm{d}\nu, \text{ when } \rho(t, x) = 0.$$

Therefore, (I, ρ, u) is a regular solution to the Cauchy problem (1.8)-(1.9) in the sense of Definition 2.1. Step 1 is proved,

To show step 2, we still need the time continuity of of u_t and $\operatorname{div} \mathbb{T}$. For u_t, it suffices to get the boundedness of $\|t^{\frac{1}{2}} \nabla^2 u_t\|_{L^\infty(0, T_*; L^2)}$ in the following lemma.

Lemma 4.1. *Let (I, φ, ϕ, u) be the unique regular solution to Cauchy problem (3.2). Then it holds that for $0 \leq t \leq T_*$,*

$$t|\nabla^2 u_t|_2^2 + \int_0^t s|\varphi \nabla^3 u_t|_2^2 ds \leq C, \tag{4.3}$$

where C is a constant depending on $A, \alpha, \beta, \gamma, \delta, T_*$.

Proof. Differentiating $(3.1)_4$ with respect to time t, we get

$$\begin{aligned} u_{tt} + \varphi^2 L u_t = &- 2\varphi \varphi_t L u - (u \cdot \nabla u)_t - \frac{\gamma - 1}{2} \nabla \left(\phi^2\right)_t + \left(\nabla \varphi^2 \cdot Q(u)\right)_t \\ &- \frac{1}{c} \int_0^\infty \int_{S^2} \left(\overline{A}_r\right)_t \Omega \mathrm{d}\Omega \mathrm{d}\nu. \end{aligned} \tag{4.4}$$

Applying the operator $\partial_x^\zeta(|\zeta|=2)$ to (4.4), multiplying by $\partial_x^\zeta u_t$ over $\mathbb{R}^3$, and integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\partial_x^\zeta u_t|_2^2+\alpha|\varphi\nabla(\partial_x^\zeta u_t)|_2^2+(\alpha+\beta)|\varphi\,\mathrm{div}(\partial_x^\zeta u_t)|_2^2\\
=&-\int\nabla\varphi^2\cdot\Big(\alpha\nabla(\partial_x^\zeta u_t)+(\alpha+\beta)\,\mathrm{div}(\partial_x^\zeta u_t)\mathbb{I}_3\Big)\cdot\partial_x^\zeta u_t\mathrm{d}x\\
&-\int\Big(\partial_x^\zeta(\varphi^2Lu_t)-\varphi^2L(\partial_x^\zeta u_t)\Big)\cdot\partial_x^\zeta u_t\mathrm{d}x\\
&-2\int\partial_x^\zeta(\varphi\varphi_tLu)\cdot\partial_x^\zeta u_t\mathrm{d}x-\int\partial_x^\zeta(u\cdot\nabla u)_t\cdot\partial_x^\zeta u_t\mathrm{d}x-\frac{\gamma-1}{2}\int\partial_x^\zeta\nabla\left(\phi^2\right)_t\cdot\partial_x^\zeta u_t\mathrm{d}x\\
&+\int\partial_x^\zeta(\nabla\varphi^2\cdot Q(u))_t\cdot\partial_x^\zeta u_t\mathrm{d}x-\frac{1}{c}\int\int_0^\infty\int_{S^2}\partial_x^\zeta(\overline{A}_r)_t\Omega\cdot\partial_x^\zeta u_t\mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x\\
\triangleq&\sum_{i=22}^{28}J_i,
\end{aligned}\tag{4.5}$$

with the natural correspondence for $J_i(i=22,23,\cdots,28)$. Using Hölder's inequality and Young's inequality, we first estimate $J_i(i=22,23,\cdots,27)$ one by one as follows:

$$\begin{aligned}
J_{22}\le&C|\varphi\nabla\partial_x^\zeta u_t|_2|\nabla\varphi|_\infty|\nabla^2u_t|_2\le\frac{\alpha}{32}|\varphi\nabla\partial_x^\zeta u_t|_2^2+C|\nabla\varphi|_\infty^2|\nabla^2u_t|_2^2,\\
J_{23}\le&C\left(|\nabla\varphi|_\infty^2|\nabla^2u_t|_2+|\varphi\nabla^2u_t|_6|\nabla^2\varphi|_3+|\varphi\nabla^3u_t|_2|\nabla\varphi|_\infty\right)|\nabla^2u_t|_2\\
\le&\frac{\alpha}{32}|\varphi\nabla^3u_t|_2^2+C\left(1+|\nabla^2u_t|_2^2\right),\\
J_{24}\le&C\Big(|\nabla^2u_t|_2\left(\|\nabla\varphi\|_2\|\varphi_t\|_2\|\nabla^2u\|_1+|\varphi\nabla^3u|_6|\nabla\varphi_t|_3+|\varphi\nabla^4u|_2|\varphi_t|_\infty\right)\\
&\quad+|\nabla^2u|_3|\nabla^2\varphi_t|_2|\varphi\nabla^2u_t|_6\Big)\\
\le&\frac{\alpha}{32}|\varphi\nabla^3u_t|_2^2+C\left(|\nabla^2u_t|_2^2+|\varphi\nabla^4u|_2^2+1\right),\\
J_{25}\le&C\left(\|u_t\|_2\|\nabla u\|_2+|\nabla u|_\infty|\nabla^2u_t|_2\right)|\nabla^2u_t|_2\le C\left(1+|\nabla^2u_t|_2^2\right),\\
J_{26}=&-\frac{\gamma-1}{2}\int\partial_x^\zeta\nabla(\phi^2)_t\partial_x^\zeta u_t\mathrm{d}x\\
\le&C\int\Big((|\nabla^3\phi||\phi_t|+|\nabla^2\phi||\nabla\phi_t|+|\nabla\phi||\nabla^2\phi_t|)|\nabla^2u_t|+|\nabla^2\phi_t||\phi\nabla^3u_t|\Big)\mathrm{d}x\\
\le&C\left((|\phi_t|_2|\nabla^3\phi|_2+|\nabla^2\phi|_6|\nabla\phi_t|_3+|\nabla\phi|_\infty|\nabla^2\phi_t|_2)|\nabla^2u_t|_2+|\nabla^2\phi_t|_2|\phi\nabla^3u_t|_2\right)\\
\le&C\left(|\varphi\nabla^3u_t|_2|\varphi|_\infty^{\frac{\gamma-\delta}{\delta-1}}|\nabla^2\phi_t|_2+|\nabla^2u_t|_2\right)\\
\le&\frac{\alpha}{32}|\varphi\nabla^3u_t|_2^2+C\left(|\nabla^2u_t|_2^2+1\right),\\
J_{27}=&\int\Big(\partial_x^\zeta(\nabla\varphi^2\cdot Q(u)_t)+\partial_x^\zeta(\nabla(\varphi^2)_t\cdot Q(u))\Big)\cdot\partial_x^\zeta u_t\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
\le & C\Bigg(\|\nabla\varphi\|_2^2|\nabla^2 u_t|_2^2 + |\nabla^2 u_t|_2\Big(\|\nabla\varphi\|_2(|\varphi\nabla^3 u_t|_2 + \|\varphi_t\|_2|\nabla^3 u|_2) + |\varphi\nabla^3 u|_6\|\varphi_t\|_2\Big) \\
& \qquad + |\varphi\nabla^2 u_t|_6\left(\|\nabla\varphi\|_2|\nabla u_t|_3 + |\nabla^3 u|_2\|\varphi_t\|_2\right)\Bigg) + \int \partial_x^\zeta(\nabla\varphi^2)_t\cdot Q(u)\cdot\partial_x^\zeta u_t \mathrm{d}x \\
\le & \frac{\alpha}{16}|\varphi\nabla^3 u_t|_2^2 + C\left(1 + |\nabla^2 u_t|_2^2 + |\varphi\nabla^4 u|_2^2\right),
\end{aligned}
$$

where the term $\int \partial_x^\zeta(\nabla\varphi^2)_t\cdot Q(u)\cdot\partial_x^\zeta u_t \mathrm{d}x$ was estimated, after integrating by parts, as

$$
\begin{aligned}
& \int \partial_x^\zeta(\nabla\varphi^2)_t\cdot Q(u)\cdot\partial_x^\zeta u_t \mathrm{d}x \\
\le & C|\nabla^2 u_t|_2|\nabla u|_\infty\|\varphi_t\|_2\|\nabla\varphi\|_2 + C\int \varphi\partial_x^\zeta\nabla\varphi_t\cdot Q(u)\cdot\partial_x^\zeta u_t \mathrm{d}x \\
\le & C|\nabla^2 u_t|_2|\nabla u|_\infty\|\varphi_t\|_2\|\nabla\varphi\|_2 + C|\varphi\nabla^3 u_t|_2|\nabla\varphi|_2|\nabla u|_\infty \\
& + C|\nabla^2\varphi_t|_2|\nabla\varphi|_\infty|\nabla^2 u_t|_2 + C|\nabla^2\varphi_t|_2|\varphi\nabla^2 u_t|_6|\nabla^2 u|_3 \\
\le & \frac{\alpha}{32}|\varphi\nabla^3 u_t|_2^2 + C(1 + |\nabla^2 u_t|_2^2).
\end{aligned}
$$

For J_{28}, by using the Hypothesis **H1** and **H2**, we have

$$
\begin{aligned}
J_{28} = & -\frac{1}{c}\int\int_0^\infty\int_{S^2}\Big(\partial_x^\zeta(\overline{\sigma}_e)_t - (\overline{\sigma}_a)_t\partial_x^\zeta I\Big)\Omega\cdot\partial_x^\zeta u_t \mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x \\
& + \frac{1}{c}\int\int_0^\infty\int_{S^2}\left(\overline{\sigma}_a + \int_0^\infty\int_{S^2}\overline{\sigma}'_s \mathrm{d}\Omega'\mathrm{d}\nu'\right)\partial_x^\zeta I_t\Omega\cdot\partial_x^\zeta u_t \mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x \\
& + \frac{1}{c}\int\int_0^\infty\int_{S^2}\Big(\partial_x^\zeta(\overline{\sigma}_a I)_t - (\overline{\sigma}_a\partial_x^\zeta I)_t\Big)\Omega\cdot\partial_x^\zeta u_t \mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x \\
& - \frac{1}{c}\int\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\frac{\nu}{\nu'}\overline{\sigma}_s\partial_x^\zeta I'_t \mathrm{d}\Omega'\mathrm{d}\nu'\Omega\cdot\partial_x^\zeta u_t \mathrm{d}\Omega\mathrm{d}\nu\mathrm{d}x \\
\le & C|\nabla^2 u_t|_2\Big(\|\nabla^2(\overline{\sigma}_e)_t\|_{L^1(\mathbb{R}^+\times S^2;L^2)} + \|(\overline{\sigma}_a)_t\|_{L^2(\mathbb{R}^+\times S^2;L^\infty)}\|\nabla^2 I\|_{L^2(\mathbb{R}^+\times S^2;L^2)} \\
& + \|\nabla^2 I_t\|_{L^2(\mathbb{R}^+\times S^2;L^2)} + \|\nabla^2(\overline{\sigma}_a)_t\|_{L^2(\mathbb{R}^+\times S^2;L^\infty)}\|I\|_{L^2(\mathbb{R}^+\times S^2;L^\infty)} \\
& + \|\nabla^2\overline{\sigma}_a\|_{L^2(\mathbb{R}^+\times S^2;L^6)}\|I_t\|_{L^2(\mathbb{R}^+\times S^2;L^\infty)} + |\nabla\overline{\sigma}_a|_{L^2(\mathbb{R}^+\times S^2;L^\infty)}\|\nabla I_t\|_{L^2(\mathbb{R}^+\times S^2;L^2)} \\
& + |\nabla(\overline{\sigma}_a)_t\|_{L^2(\mathbb{R}^+\times S^2;L^6)}\|\nabla I\|_{L^2(\mathbb{R}^+\times S^2;L^3)}\Big) \\
\le & C|\nabla^2 u_t|_2\Big(\|(\overline{\sigma}_e)_t\|_{L^1(\mathbb{R}^+\times S^2;H^2)} + \|(\overline{\sigma}_a)_t\|_{L^2(\mathbb{R}^+\times S^2;H^2)}\|I\|_{L^2(\mathbb{R}^+\times S^2;H^2)} \\
& \qquad + (1 + \|\nabla\overline{\sigma}_a\|_{L^2(\mathbb{R}^+\times S^2;H^2)})\|I_t\|_{L^2(\mathbb{R}^+\times S^2;H^2)}\Big) \\
\le & C\left(1 + |\nabla^2 u_t|_2^2\right).
\end{aligned}
$$

Substituting $J_i(i = 22, \cdots, 28)$ into (4.5) and and summing up all $|\zeta| = 2$, we get

$$\frac{d}{dt}|\nabla^2 u_t|_2^2 + \alpha|\varphi\nabla^3 u_t|_2^2 \leq C(|\nabla^2 u_t|_2^2 + |\varphi\nabla^4 u|_2^2 + 1). \tag{4.6}$$

Multiplying (4.6) by s and integrating with respect to s over $[\tau, t]$ for $\tau \in (0, t)$, it yields

$$t|\nabla^2 u_t|_2^2 + \alpha \int_\tau^t s|\varphi\nabla^3 u_t|_2^2 \mathrm{d}s \leq \tau|\nabla^2 u_t|_2^2 + C. \tag{4.7}$$

Recalling the definition of regular solutions, we have

$$\nabla^2 u_t \in L^2([0, T_*]; L^2), \tag{4.8}$$

which, together with (4.4) yields

$$u_{tt} \in L^2([0, T_*]; L^2). \tag{4.9}$$

Using Lemma 2.6, we know that there exists a sequence s_k such that

$$s_k \to 0, \quad \text{and} \quad s_k|\nabla^2 u_t(\cdot, s_k)|_2^2 \to 0, \quad \text{as}\, k \to +\infty.$$

After choosing $\tau = s_k \to 0$ to (4.7), we conclude that

$$t|\nabla^2 u_t|_2^2 + \int_0^t s|\varphi\nabla^3 u_t|_2^2 \mathrm{d}s \leq C. \tag{4.10}$$

The proof of Lemma 4.1 is finished. □

Now we continue the proof of step 2. Using the Sobolev embedding

$$L^2([0, T_*]; H^1) \cap W^{1,2}([0, T_*]; H^{-1}) \hookrightarrow C([0, T_*]; L^q), \quad q \in (3, 6], \tag{4.11}$$

we obtain from Lemma 4.1 that

$$tu_t \in C([0, T_*]; W^{1,4}),$$

which follows $u_t \in C((0, T_*] \times \mathbb{R}^3)$.

Since $\operatorname{div} \mathbb{T} = \rho\mathbb{H}$, where $\mathbb{H} = \varphi^2 Lu - \nabla\varphi^2 \cdot Q(u)$, it suffices to prove the time continuity of $\mathbb{H}$. According to $(3.1)_4$, we know that

$$t\mathbb{H} \in L^\infty([0, T_*]; H^2).$$

From the fact that $\mathbb{H}_t \in L^2([0, T_*]; L^2)$ and (4.11), it follow

$$t\mathbb{H} \in C([0, T_*] \times \mathbb{R}^3),$$

which implies $\mathbb{H} \in C((0, T_*] \times \mathbb{R}^3)$ and thus the time continuity of $\operatorname{div} \mathbb{T}$, which, together with (4.1)-(4.2), finishes the proof of step 2. Theorem 2.1 is proved.

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