Asymptotical mean-square stability of linear θ-methods for stochastic pantograph differential equations: variable stepsize and transformation approach

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May 22, 2020

Abstract

The paper deals with the asymptotical mean-square stability of the linear θ-methods under variable stepsize and transformation approach for stochastic pantograph differential equations. A limiting equation for the analysis of numerical stability is introduced by Kronecker products. Under the condition which guarantee the stability of exact solutions, the optimal stability region of the linear θ-methods under variable stepsize is given by using the limiting equation, i.e., $\theta \in (\frac{1}{2}, 1]$, which is the same to the deterministic problems. Moreover the linear θ-methods under the transformation approach are also considered and the result of the stability is improved for $\theta = \frac{1}{2}$. Finally, numerical examples are given to illustrate the asymptotical mean-square stability under variable stepsize and transformation approach.

Keywords. Stochastic pantograph differential equations, Linear θ-methods, Asymptotical mean-square stability, Variable stepsize, Transformation approach

1 Introduction

Stochastic differential systems, such as stochastic differential equations (SDEs), stochastic delay differential equations (SDDEs), have attracted much attention due to their applications in physical, dynamical systems, and finance. Stochastic pantograph differential equations (SPDEs) are a special kind of Itô-type SDDEs with unbounded memory and a generalization of deterministic pantograph equations, i.e.,

$$\begin{cases} x'(t) = ax(t) + bx(qt), & t \geq 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $0 < q < 1$, and $x_0 \in \mathbb{R}$, a, b are constants. In recent years, many authors have considered SPDEs

$$\begin{cases} \mathrm{d}x(t) &= [ax(t) + bx(qt)]\mathrm{d}t + [cx(t) + dx(qt)]\mathrm{d}W(t), \quad t > 0, \\ x(0) &= x_0, \end{cases} \tag{1.2}$$

where $0 < q < 1$, a, b, c, d are constants, $x_0 \in \mathbb{R}$ and W(t) is a standard Wiener process.

Many important results have been discussed for the exact solutions. Baker and Bukwar [1] gave the necessary analytical theory for existence and uniqueness of a strong solution. Fan et al. [4] gave the Razumikhin-type theorems of the αth moment asymptotical stability of exact solutions for linear stochastic pantograph differential equations. Guo and Li [7] established Razumikhin-type theorems on the αth moment polynomial stability of exact solutions for the nonlinear stochastic pantograph differential equations. Yang et al. [25] investigated the mean-square stability of nonlinear stochastic pantograph differential equations, and an equivalent form of stochastic delay differential equations with constant delay is introduced by the transformation approach.

Many authors investigated numerical solutions with constant stepsize to SPDEs [3, 4]. Because the most difficult problem is the limited computer memory, many authors applied variable stepsize and transformation approach for the deterministic pantograph equations to solve the storage problem. Recently, the linear θ-methods under variable stepsize have been discussed for SPDEs in [22, 23], of which the strong order of convergence $p = \frac{1}{2}$ is given under the Lipschitz condition in [23] and the mean-square stability is studied in [22]. Whereas, the stability region of θ in [22], i.e., $\theta \in (\frac{|a|+|b|}{2|a|}, 1]$, is a little stronger than deterministic problems in the sense that for $c, d = 0$, the equation (1.2) is equivalent to the deterministic pantograph equations. Therefore, we are interested in an optimal stability region of the linear θ-methods.

For deterministic pantograph equations, many authors investigated the numerical stability by the limiting equation, such as Liu [14], Xu [24], Liu [17], Wang [20] and so on, which always provides an optimal stability result. Up to the best of our knowledge, there is no such work for SPDEs. In order to introduce the limiting equation for SPDEs, we, instead of the inner product, employ the Kronecker product to obtain a matrix equation. Such a technique has been used in [21] for the asymptotical mean-square stability and in [27] for the numerical stability. However, for SPDEs, the coefficients in the matrix equation varies with the numerical process. Hence a limiting equation is also in great need to introduce and an optimal stability region of linear θ-methods is obtained for linear SPDEs.

Another approach, i.e., transformation approach, is also widely used to overcome the storage problems for deterministic pantograph equations. Moreover, there are some stability results under such an approach. Koto [13] investigated the stability of Runge-Kutta methods. Liu [14] proved that the linear θ-method is Λ-stable. However, up to the best of our knowledge, there is no such work of the numerical solutions for SPDEs. Yang et al. [25] proposed SPDEs is equivalent to SDDEs with constant delay by the transformation approach. Hence we will discuss the asymptotical mean-square stability of the linear θ-methods applying to the differential form proposed in [25]. By using limiting equations again, we get the main result of this paper, that is the stability region is the same to the deterministic problems [14], i.e., $\theta \in [\frac{1}{2}, 1]$.

The paper is organized as follows. In section 2, we gave some preliminaries. In section 3, we investigate the asymptotical mean-square stability under variable stepsize for linear SPDEs if $\theta \in (\frac{1}{2}, 1]$. In section 4, the asymptotical mean-square stability is improved by transformation approach, i.e., $\theta \in [\frac{1}{2}, 1]$. We will provide some numerical examples in the section 5.

2 Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_{t\geq 0}\}$ satisfying the usual conditions. By the definition of stochastic differential, the equation (1.2) can be expressed equivalently as

$$x(t) = x_0 + \int_0^t ax(s) + bx(qs)\mathrm{d}s + \int_0^t cx(s) + dx(qs)\mathrm{d}W(s).$$

Definition 2.1. [22] An $\mathbb{R}$-valued stochastic process $x(t) : [0, T] \times \Omega \to \mathbb{R}$ is called a strong solution of equation (1.2), if it is a measurable, sample-continuous process such that $x|_{[0,T]}$ is $(\mathcal{F}_t)_{0\leq t\leq T}-$adapted, and x satisfies equation (1.2), almost surely, and satisfies the initial condition $x(0) = x_0$. A solution $x(t)$ is said to be path-wise unique if any other solution $\widehat{x}(t)$ is stochastically indistinguishable from it, i.e.

$$P\{x(t) = \widehat{x}(t), \quad for\ all\ 0 \leq t \leq T\} = 1.$$

Lemma 2.2. [1] If $0 < q < 1$ and $E|x_0|^2 < \infty$, then there exists a path-wise unique strong solution to problem (1.2).

Lemma 2.3. [4] If the coefficient of equation (1.2) are satisfied

$$a < -|b| - \frac{1}{2}(|c| + |d|)^2, \tag{2.1}$$

then the trivial solution is asymptotically mean-square stable, that is

$$\lim_{t\to\infty} E|x(t)|^2 = 0.$$

For the convenience of discussion in the following sections, we review some matrix knowledge in [9].

Definition 2.4. The Kronecker product of $A = [a_{ij}] \in \mathbb{C}_{m,n}$ and $B = [b_{ij}] \in \mathbb{C}_{p,q}$ is denoted by $A \otimes B$ and is defined to the block matrix

$$A \otimes B \equiv \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{pmatrix} \in \mathbb{C}_{mp,nq}.$$

Definition 2.5. With each matrix $A = [a_{ij}] \in \mathbb{C}_{m,n}$, we associate the vector $vecA \in \mathbb{C}_{m,n}$ defined by

$$vecA \equiv [a_{11}, \cdots, a_{m1}, a_{12}, \cdots, a_{m2}, \cdots, a_{1n}, \cdots, a_{mn}]^T.$$

Lemma 2.6. Let $A \in \mathbb{C}_{m,n}$, $B \in \mathbb{C}_{p,q}$, $C \in \mathbb{C}_{m,q}$ be given and let $X \in \mathbb{C}_{n,p}$ be unknown. The matrix equation $AXB = C$ is equivalent to the system of qm equation in np unknown given by $vec(AXB) = (B^T \otimes A)vecX$.

Lemma 2.7. Let $A \in \mathbb{C}_n$, $B \in \mathbb{C}_m$, and the point set of eigenvalues be denoted by $\sigma(\cdot)$. If $\lambda \in \sigma(A)$ and $\mu \in \sigma(B)$, then $\lambda\mu \in \sigma(A \otimes B)$. Every eigenvalue of $A \otimes B$ arises as such a product of eigenvalues of A and B. If $\sigma(A) = \{\lambda_1, \cdots, \lambda_n\}$, and $\sigma(B) = \{\mu_1, \cdots, \mu_m\}$, then $\sigma(A \otimes B) = \{\lambda_i\mu_j : i = 1, \cdots, n, j = 1, \cdots, m\}$.

3 The linear θ-methods under variable stepsize

For the convenience of readers, we review some background information of the linear θ-methods under variable stepsize for (1.2) in [22] as follows:

Here, the mesh $H = \{m; t_0, t_1 \cdots, t_n \cdots\}$ is defined as follows. Let $T_0 > 0$ be given, $t_0 = T_0$ and $t_m = q^{-1}T_0$. We choose $m - 1$ grid points $t_1 < t_2 < \cdots < t_{m-1}$ in (t_0, t_m) and define other points by

$$t_{km+i} = q^{-k}t_i, \text{ for } k = 1, 2, \cdots, \ i = 0, 1, \cdots m-1.$$

It is easy to see that the grid points t_n satisfy $qt_n = t_{n-m}$ for $n \geq 0$ and the variable stepsize $h_n = t_{n+1} - t_n$ satisfies

$$h_n = q^{-1}h_{n-m}, \text{ for all } n \geq 1, \text{ and } \lim_{n\to\infty} h_n = \infty.$$

Furthermore, we suppose to have numerical solutions available till the point T_0, which is called initial data.

The linear θ-methods under variable stepsize of the equation (1.2) have the form

$$\begin{aligned} x_{n+1} =& x_n + \theta h_n(ax_{n+1} + bx_{n-m+1}) + (1-\theta)h_n(ax_n + bx_{n-m}) \\ &+ (cx_n + dx_{n-m})\Delta W_n, \end{aligned} \tag{3.1}$$

where x_n is an approximation to $x(t_n)$, θ is a parameter with $0 \leq \theta \leq 1$ and the increments $\Delta W_n := W(t_{n+1}) - W(t_n)$ are independent $N(0, h_n)$-distributed Gaussian random variables. Further, x_n is $\mathcal{F}_{t_n}$-measurable at the mesh-point t_n. Since the strong convergence of numerical methods with variable stepsize for linear SPDEs has been studied by Xiao et al. [22], we will consider the asymptotical mean-square stability.

Definition 3.1. The linear θ-methods with variable stepsize for (1.2) are said to be asymptotical mean-square stability, if for any given mesh H

$$\lim_{n\to\infty} E|x_n|^2 = 0.$$

From (3.1), we obtain

$$\begin{aligned} (1 - a\theta h_n)x_{n+1} =& (1 + (1-\theta)ah_n)x_n + \theta h_n b x_{n-m+1} \\ &+ (1-\theta)h_n b x_{n-m} + (cx_n + dx_{n-m})\Delta W_n, \end{aligned} \tag{3.2}$$

which has an equivalent form

$$M_{0n}X_{n+1} = M_{1n}X_n + M_{2n}X_n\Delta W_n,$$

where $X_n = (x_n, x_{n-1}, \cdots, x_{n-m})^T$,

$$M_{0n} = \begin{pmatrix} 1 - a\theta h_n & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$M_{1n} = \begin{pmatrix} 1+(1-\theta)ah_n & 0 & \cdots & 0 & b\theta h_n & b(1-\theta)h_n \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$M_{2n} = \begin{pmatrix} c & 0 & \cdots & 0 & d \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

It follows from (2.1) that $1 - a\theta h_n \neq 0$, which yields that

$$X_{n+1} = M_1^n X_n + \widetilde{M_2^n} X_n \Delta W_n,$$

where

$$M_1^n = M_{0n}^{-1} M_{1n} = \begin{pmatrix} \frac{1+(1-\theta)ah_n}{1-a\theta h_n} & 0 & \cdots & 0 & \frac{b\theta h_n}{1-a\theta h_n} & \frac{b(1-\theta)h_n}{1-a\theta h_n} \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \qquad (3.3)$$

$$\widetilde{M_2^n} = M_{0n}^{-1} M_{2n} = \begin{pmatrix} \frac{c}{1-a\theta h_n} & 0 & \cdots & 0 & \frac{d}{1-a\theta h_n} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Remark 3.2. Some sufficient conditions for the mean-square stability of the linear θ-methods has been studied by the inner produce, since $\lim_{n\to\infty} E|x_n|^2 = 0$ is equivalent to $\lim_{n\to\infty} E|X_n^T X_n| = 0$. However, we are interested in an optimal stability region of the linear θ-methods for linear SPDEs. In view of the technique in [21] for the asymptotical mean-square stability and for the numerical stability in [27], we replace the inner product $E|X_n^T X_n|$ by the equivalent kronecker product $E|X_n X_n^T|$.

Noting that $E \mid \Delta W_n \mid = 0$, $E \mid \Delta W_n \mid^2 = h_n$ and X_n, X_{n-m+1}, X_{n-m} are $\mathcal{F}_{t_n}-$measurable, hence

$$\begin{aligned} E(X_n X_n^T \Delta W_n) &= E[X_n X_n^T E(\Delta W_n | \mathcal{F}_{t_n})] = 0, \\ E(X_n X_n^T (\Delta W_n)^2) &= E[X_n X_n^T E((\Delta W_n)^2 | \mathcal{F}_{t_n})] = h_n E(X_n X_n^T). \end{aligned}$$

Thus, we have

$$E(X_{n+1}X_{n+1}^T) = M_1^n E(X_n X_n^T)(M_1^n)^T + h_n^{\frac{1}{2}} \widetilde{M_2^n} E(X_n X_n^T)(\widetilde{M_2^n})^T (h_n^{\frac{1}{2}})^T, \quad (3.4)$$

which with the notations $D_{n+1} = E(X_{n+1}X_{n+1}^T)$ and $h_n^{\frac{1}{2}} \widetilde{M_2^n} = M_2^n$ is rewritten as

$$D_{n+1} = M_1^n D_n (M_1^n)^T + M_2^n D_n (M_2^n)^T.$$

Hence, in view of Lemma 2.6, we obtain that

$$vec(D_{n+1}) = A_n vec(D_n), \quad (3.5)$$

with $A_n = M_1^n \otimes M_1^n + M_2^n \otimes M_2^n$.

Remark 3.3. Since Liu [14] proposed the limiting equation for deterministic equations, many authors investigated the stability by the limiting equation. Here, we introduce a limiting equation to obtain an optimal stability region of linear θ-methods in this paper. In view of $\lim_{n\to\infty} h_n = \infty$, ones obtain that

$$\lim_{n\to\infty} M_1^n = M_1 = \begin{pmatrix} -\frac{1-\theta}{\theta} & 0 & \cdots & 0 & -\frac{b}{a} & -\frac{b(1-\theta)}{a\theta} \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \quad (3.6)$$

$$\lim_{n\to\infty} M_2^n = M_2 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Hence the limit of the matrix A_n exists, which is given by $A = M_1 \otimes M_1$, and the limiting equation of (3.5) is defined as

$$Z_{n+1} = AZ_n. \quad (3.7)$$

It is ready to formulate our main results that, under the condition (2.1), which is independent of q, the asymptotical mean-square stability condition of the linear θ-methods is the same as the deterministic problem in the following theorem.

Theorem 3.4. Under the condition (2.1), if $\theta \in (\frac{1}{2}, 1]$, then the linear θ-methods with variable stepsize of the equation (1.2) are asymptotical mean-square stability, i.e.,

$$\lim_{n\to\infty} E|x_n|^2 = 0.$$

Proof. From the Remark 3.2, it is easy to see that $\lim_{n\to\infty} E|x_n|^2 = 0$ is equivalent to $\lim_{n\to\infty} D_n = 0$. Referring to [17] we can know that $\lim_{n\to\infty} Z_n = 0$ implies $\lim_{n\to\infty} D_n = 0$, if two conditions are satisfied, i.e., $\Sigma_{n=0}^{\infty}\|A_n - A\|_\infty < \infty$ and the algebraic multiplicity of the eigenvalue for A is 1.

The problem turns out to verify the conditions in [17]. Applying the Cauchy radical test to (3.3) and (3.6), we know

$$\begin{aligned}\Sigma_{n=0}^{\infty}\left|\frac{1+(1-\theta)ah_n}{1-a\theta h_n}+\frac{1-\theta}{\theta}\right| &= \Sigma_{n=0}^{\infty}\left|\frac{1}{(1-a\theta h_n)\theta}\right| < \infty,\\ \Sigma_{n=0}^{\infty}\left|\frac{b\theta h_n}{1-a\theta h_n}+\frac{b}{a}\right| &= \Sigma_{n=0}^{\infty}\left|\frac{1}{(1-a\theta h_n)\theta}\right| < \infty,\\ \Sigma_{n=0}^{\infty}\left|\frac{b(1-\theta)h_n}{1-a\theta h_n}+\frac{b(1-\theta)}{a\theta}\right| &= \Sigma_{n=0}^{\infty}\left|\frac{1}{(1-a\theta h_n)\theta}\right| < \infty,\end{aligned}$$

which yields $\Sigma_{n=0}^{\infty}\|M_1^n - M_1\|_\infty < \infty$. By a similar calculation, $\Sigma_{n=0}^{\infty}\|M_2^n - M_2\|_\infty < \infty$. Hence from the properties of Kronecker product of matrix, we obtain $\Sigma_{n=0}^{\infty}\|A_n - A\|_\infty < \infty$.

For the eigenvalue for A, we only need to consider the eigenvalues of M_1, which is given by the roots of the equations

$$|\lambda I - M_1| = (\lambda + \tfrac{1-\theta}{\theta})(\lambda^m + \tfrac{b}{a}) = 0,$$

where I is the identity matrix of order $m+1$. Namely the eigenvalues are

$$\lambda_m = \tfrac{\theta-1}{\theta},\ \lambda_k = \sqrt[m]{|\tfrac{b}{a}|}e^{i\frac{1}{m}(\arg(-\frac{b}{a})+2k\pi)},\ k = 0, \cdots, m-1.$$

Therefore, by Lemma 2.7, the algebraic multiplicity of the eigenvalue for A is 1.

It follows from condition (2.1) and $\theta \in (\frac{1}{2}, 1]$ that $\rho(M_1) < 1$, which together with Lemma 2.7 implies that $\rho(A) < 1$. Hence the proof is complete. $\square$

From Theorem 3.4, the stability region of linear θ-methods is the same as the deterministic problems, i.e., $\theta \in (\frac{1}{2}, 1]$. Furthermore, the stability region of θ is improved by the transformation approach for deterministic pantograph equations. Hence, we will investigated asymptotical mean-square stability under transformation approach in the next section.

4 The linear θ-methods under transformation approach

In this section, we are interested in the asymptotical mean-square stability under transformation approach. Applying the time-scale transformation approach to (1.2) by $y(t) = x(e^t)$, the stochastic process $y(t)$ satisfies stochastic delay differential equations with a constant delay

$$\begin{cases} \mathrm{d}y(t) & = e^t[ay(t) + by(t-\tau)]\mathrm{d}t + e^{\frac{1}{2}t}[cy(t) + dy(t-\tau)]\mathrm{d}B(t), \quad t \geq t_0, \\ y(0) & = x(e^t), \quad t \leq t_0, \end{cases} \tag{4.1}$$

where $\tau = -\ln q$ and $W(e^t)$, the time-changed Wiener process, has a differential form $\mathrm{d}W(e^t) = e^{\frac{1}{2}t}\mathrm{d}B(t)$ with a standard Brownian motion $B(t)$ with respect to the filtration $\{\mathcal{F}_{e^t}\}_{t \geq t_0}$. Hence, the linear θ-methods under the transformation approach of the equation (4.1) have the form:

$$\begin{aligned} y_{n+1} = & y_n + \theta h e^{t_{n+1}}(ay_{n+1} + by_{n-m+1}) \\ & + (1-\theta)he^{t_n}(ay_n + by_{n-m}) + e^{\frac{1}{2}t_n}(cy_n + dy_{n-m})\Delta B_n, \end{aligned} \tag{4.2}$$

where $h = -\frac{\ln q}{m}$, $\theta \in [0,1]$, y_n is an approximation to $y(t_n)$ and the increments $\Delta B_n := B(t_{n+1}) - B(t_n)$ are independent $N(0,h)$-distributed Gaussian random variables. Moreover, we assume that y_n is $\mathcal{F}_{e^{t_n}}$-measurable at the mesh-point t_n.

Remark 4.1. The strong convergence of (4.2) comes directly from the discussion in [19] for the linear θ-methods for SDDEs with a variable coefficients. However, the stability analysis is much harder, since the one-sides condition and linear growth condition are hardly satisfied at the same time.

Definition 4.2. The linear θ-methods under transformation approach are said to be asymptotically stable in the mean-square sense, if $\lim\limits_{n\to\infty} E|y_n|^2 = 0$ holds for any $h > 0$, where y_n is the solution of (4.2).

From (4.2), we can get

$$\begin{aligned} (1 - a\theta h e^{t_{n+1}})y_{n+1} = & (1 + (1-\theta)ahe^{t_n})y_n + \theta h e^{t_{n+1}} b y_{n-m+1} \\ & + (1-\theta)he^{t_n} b y_{n-m} + e^{\frac{1}{2}t_n}(cy_n + dy_{n-m})\Delta B_n, \end{aligned} \tag{4.3}$$

which has an equivalent form

$$N_{0n}Y_{n+1} = N_{1n}Y_n + N_{2n}Y_n\Delta B_n,$$

where $Y_n = (y_n, y_{n-1}, \cdots, y_{n-m+1}, y_{n-m})^T$,

$$N_{0n} = \begin{pmatrix} 1 - a\theta h e^{t_{n+1}} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$N_{1n} = \begin{pmatrix} 1 + (1-\theta)ahe^{t_n} & 0 & \cdots & 0 & b\theta h e^{t_{n+1}} & b(1-\theta)he^{t_n} \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$N_{2n} = \begin{pmatrix} ce^{\frac{1}{2}t_n} & 0 & \cdots & 0 & de^{\frac{1}{2}t_n} \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

It follows from (2.1) that $1 - a\theta h e^{t_{n+1}} \neq 0$, which yields that

$$Y_{n+1} = N_1^n Y_n + \widetilde{N}_2^n Y_n \Delta B_n,$$

where

$$N_1^n = N_{0n}^{-1} N_{1n} = \begin{pmatrix} \frac{1+(1-\theta)ahe^{t_n}}{1-a\theta h e^{t_{n+1}}} & 0 & \cdots & 0 & \frac{b\theta h e^{/t_{n+1}}}{1-a\theta h e^{t_{n+1}}} & \frac{b(1-\theta)he^{t_n}}{1-a\theta h e^{t_{n+1}}} \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$\widetilde{N}_2^n = N_{0n}^{-1} N_{2n} = \begin{pmatrix} \frac{ce^{\frac{1}{2}t_n}}{1-a\theta h e^{t_{n+1}}} & 0 & \cdots & 0 & \frac{de^{\frac{1}{2}t_n}}{1-a\theta h e^{t_{n+1}}} \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{4.4}$$

Noting that $E \mid \Delta B_n \mid = 0$, $E \mid \Delta B_n \mid^2 = h$, and Y_n, Y_{n-m+1}, Y_{n-m} are $\mathcal{F}_{t_n}-$measurable, hence

$$E(Y_n Y_n^T \Delta B_n) = E[Y_n Y_n^T E(\Delta B_n | \mathcal{F}_{t_n})] = 0,$$
$$E(Y_n Y_n^T (\Delta B_n)^2) = E[Y_n Y_n^T E((\Delta B_n)^2 | \mathcal{F}_{t_n})] = hE(Y_n Y_n^T).$$

Therefore, in view of Remark 3.2, we have

$$E(Y_{n+1}Y_{n+1}^T) = N_1^n E(Y_n Y_n^T)(N_1^n)^T + h^{\frac{1}{2}} \widetilde{N}_2^n E(Y_n Y_n^T)(\widetilde{N}_2^n)^T h^{\frac{1}{2}}.$$

Denoting $U_{n+1} = E(Y_{n+1}Y_{n+1}^T)$, $\widetilde{N}_2^n h^{\frac{1}{2}} = N_2^n$ and using Lemma 2.6, we obtain that

$$vec(U_{n+1}) = Q_n vec(U_n), \tag{4.5}$$

wheere $Q_n = N_1^n \otimes N_1^n + N_2^n \otimes N_2^n$. In view of $\lim\limits_{n\to\infty} e^{t_n} = \infty$, ones obtain that

$$\lim_{n\to\infty} N_1^n = N_1 = \begin{pmatrix} -\frac{1-\theta}{\theta e^h} & 0 & \cdots & 0 & -\frac{b}{a} & -\frac{b(1-\theta)}{a\theta e^h} \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \tag{4.6}$$

$$\lim_{n\to\infty} N_2^n = N_2 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Hence the limit of matrix Q_n exist, which is given by $Q = N_1 \otimes N_1$, By Remark 3.3, the limiting equation of (4.5) is defined as $W_{n+1} = QW_n$.

Compared with Theorem 3.4, the asymptotical mean-square stability region of linear θ-methods under the transformation approach is improved by $\theta \in [\frac{1}{2}, 1]$ in the following theorem.

Theorem 4.3. Under the condition (2.1), if $\theta \in [\frac{1}{2}, 1]$, then the linear θ-methods with transformation approach of the equation (1.2) are asymptotically stable in the mean-square sense.

Proof. Similarly to the argument of Theorem 3.4, we will verify the conditions in [17] firstly. It is easily seen that $\Sigma_{n=0}^{\infty} \|Q_n - Q\|_\infty < \infty$. For the eigenvalue of Q, we only need to consider the eigenvalues of N_1, which is given by the roots of the equations

$$|\mu I - N_1| = (\mu + \frac{1-\theta}{\theta e^h})(\mu^m + \frac{b}{a}).$$

Namely the eigenvalues are

$$\mu_m = \tfrac{\theta-1}{\theta e^h},\ \mu_k = \sqrt[m]{|\tfrac{b}{a}|} e^{i\frac{1}{m}(\arg(-\frac{b}{a})+2k\pi)},\ k = 0, \cdots, m-1.$$

Therefore, by Lemma 2.7, the algebraic multiplicity of the eigenvalue for Q is 1.

Compared with the result in Theorem 3.4, the condition is improved by $|\frac{1-\theta}{\theta e^h}| < 1$ for all $\theta \in [\frac{1}{2}, 1]$. Hence, the proof is complete. $\square$

The linear θ-methods with transformation approach are asymptotically stable in the mean-square sense for $\frac{1}{2} \leq \theta \leq 1$, which is the same to the deterministic problems. Similarly to the deterministic pantograph equations, the condition is improved by transformation approach.

5 Numerical experiments

In this section, we will present several numerical examples to illustrate the asymptotical mean-square stability under variable stepsize and the transformation approach. We consider the following linear stochastic pantograph differential equation

$$\mathrm{d}x(t) = [ax(t) + bx(qt)]\mathrm{d}t + [cx(t) + dx(qt)]\mathrm{d}W(t), \tag{5.1}$$

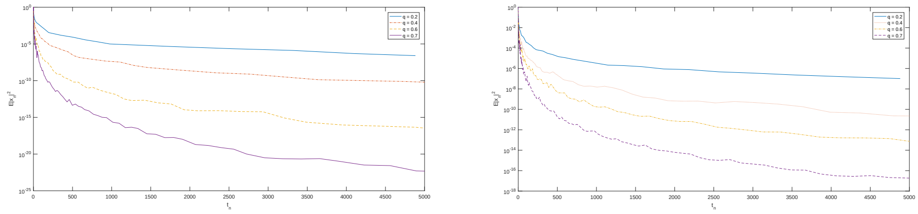
with initial condition $x(0) = 1$. The data plotted in all figures are obtained as the mean-square data for 4000 trajectories, that is $\omega_i : 1 \leq i \leq 4000$, $E|x_n|^2 = \frac{1}{4000}\Sigma_{i=1}^{4000}|x_n(\omega_i)|^2$.

Case 1: we choose the coefficients as $a = -3, b = 0.5, c = 1, d = 1, \theta = 0.6$. From figure 1, we can see that the asymptotical mean-square stability with different q is hold under variable stepsize and the transformation approach. Especially for $q = 0.4$, the coefficients satisfy the condition (2.1), but do not satisfy the condition in [26]. It is easy to see form Figure 1 that the asymptotical mean-square stability is still hold, which is coincide with Theorem 3.4 and Theorem 4.3. That is, the stability condition is independent of q under variable stepsize and the transformation approach.

Case 2: we use the set of parameters I: $a = -3, b = 0.5, c = 1, d = 1, q = 0.5$ and II: $a = -4, b = 1, c = 1, d = 1, q = 0.5$. It is easy to see that the asymptotical mean-square stability is hold with different θ in Figure 2, which is coincide with Theorem 3.4. Especially, comparing with the range of θ in [22], i.e., $\theta \in (\frac{|a|+|b|}{2|a|}, 1]$, we can easily get $0.6 \notin (\frac{|a|+|b|}{2|a|}, 1]$. However, the asymptotical mean-square stability is still hold for $\theta = 0.6$ in Figure 2 , which implies the asymptotical mean-square stability region of linear θ-methods is improved by $\theta \in (\frac{1}{2}, 1]$ under variable stepsize.

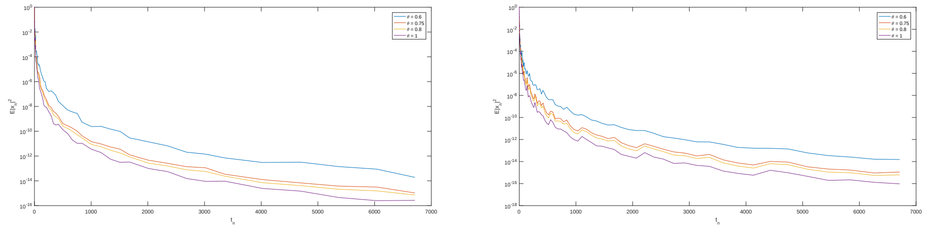
Case 3: we choose the coefficients as $a = -5, b = 1, c = 1, d = 1, q = 0.6$. Under $m = 2, 5, 10, 20$, it is easy to see from the Figure 3 that the linear

θ-methods with $\theta = \frac{1}{2}$ are asymptotical mean-square stability for transformation approach but not for variable stepsize. We can know that the condition is improved by transformation approach, which is coincide with Theorem 3.4 and Theorem 4.3. The figure implies us the condition may be necessary.



(a) variable stepsize (b) transformation approach

Figure 1: The asymptotical mean-square stability with different q



(a) $a = -3, b = 0.5, c = 1, d = 1, q = 0.5$ (b) $a = -4, b = 1, c = 1, d = 1, q = 0.5$

Figure 2: The asymptotical mean-square stability for different θ under variable stepsize

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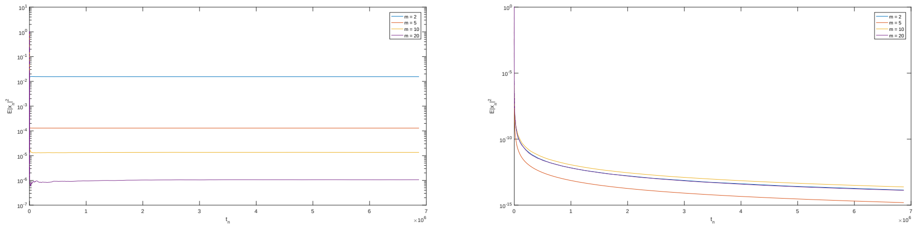

(a) variable stepsize (b) transformation approach

Figure 3: The influence of $\theta = \frac{1}{2}$ on the asymptotical mean-square stability under variable stepsize and transformation

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