A Sequential Nonlinear Random Fractional Differential Equation: Existence, Uniqueness and New Data Dependence

Hafssa Yfrah[1], Zoubir Dahmani[2],
Mehmet Zeki Sarikaya[3] and Farooq Ahmed Gujar[4]

[1,2]Laboratory of Pure and Applied Mathematics,
Faculty of Exact Sciences and Informatics, University of Mostaganem, 27000, Algeria,
[3]Art and Sciences Faculty, Department of Mathematics, Duzce University,
[4]Nanyang Technological University, Singapore.
Emails: hafssa2014yfrah@gmail.com, zzdahmani@yahoo.fr
sarikayamz@gmail.com, farooqgujar@gmail.com

Abstract

In this work, we are concerned with a sequential nonlinear random differential equation of fractional order with nonlocal conditions. This is the first time in the literature where sequential problems and random ones are combined and considered. An existence and uniqueness of solutions for the problem is obtained by means of an appropriate random fixed point theorem. Then, new concepts on the sequential continuous and fractional derivative dependence are introduced. At the end, some results of stability on random, as well for deterministic, data dependence are discussed.

Key Words: Caputo derivative, sequential random differential equation, existence and uniqueness, mean square solution.

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Introduction

Fractional calculus is appearing in the different fields of scientific research such as: applied mathematics, physics, control theory, mechanical structures, thermodynamics, etc. [2, 7, 16]. For some recent studies on fractional calculus and fractional differential equations (FDEs), we refer the reader to the papers [3, 4, 6, 14, 17, 19].
Random fractional differential equations, as natural extensions of deterministic ones, arise in numerous fields with anomalous dynamics, such as network traffic, signal transmissions through strong magnetic fields atmospheric diffusion of pollution, etc. [1, 5, 15, 20, 21]. Recently, the initial random fractional differential problems have been investigated by several authors, to cite a few, we begin by the paper [11], where the authors have studied the following interesting nonlinear random FDE with a nonlocal condition:

$$
\begin{cases}
\mathbf{D}^\alpha X(t) &= c(t)f(X(t)) + b(t),\ t \in [0,T] \\
X_0 &= X(0) + \sum_{k=1}^{n} a_k X(\tau_k),\ a_k > 0,\ \tau_k \in]0,T[.
\end{cases}
$$

where, $\mathbf{D}^\alpha$ represent the mean square Caputo fractional derivative of order $\alpha \in]0, 1]$.
Then, based on the above paper, the authors of [18] have been concerned with the following random problem

$$
\begin{cases}
\mathbf{D}^\alpha X(t) &= c(t)f(X(t)) + b(t)g\big(\mathbf{D}^{\alpha-1}X(t)\big) \\
X_0 &= X(0) + \sum_{k=1}^{n} a_k X(\tau_k) \\
X_1 &= X'(0).
\end{cases} \quad ,
$$

where, $\mathbf{D}^\alpha$ represent the mean square Caputo fractional derivative of order $\alpha \in]1, 2]$.
Very recently, the authors of the paper [22] have studied the following high order nonlinear random FDE:

$$
\begin{cases}
\mathbf{D}^\alpha X(t) &= c(t)f(X(t)) + b(t)g\big(\mathbf{D}^{\alpha-1}X(t), \ldots, \mathbf{D}^{\alpha-n+1}X(t)\big), \\
X_0 &= X(0) + \sum_{k=1}^{n} a_k X(\tau_k), \\
X_j &= X^j(0) \quad j = 1, \ldots, n-1.
\end{cases} \quad .
$$

Some recent concepts have been introduced and other data dependance of the solutions have been discussed.
In this paper, we are concerned with a new class of nonlinear random differential equations with nonlocal conditions that involves sequential mean square derivatives. So, we consider the problem:

$$
\begin{cases}
\mathbf{D}^\alpha\big(\mathbf{D}^\beta X\big)(t) &= c(t)f(X(t)) + b(t)g\big(\mathbf{D}^\beta X(t)\big), \\
X_0 &= X(0) + \sum_{k=1}^{n} a_k X(\tau_k), \, a_k > 0, \, \tau_k \in]0, T[\quad , \\
X_1 &= X'(0),
\end{cases} \quad (1)
$$

where, $\mathbf{D}^\alpha$ and $\mathbf{D}^\beta$ represents the mean square Caputo fractional derivative, with, α and β are in $]0, 1]$, $X(\cdot)$ is a second random function, X_0, X_1 are a second random variable and a_k are positive real numbers, $f : \mathbb{L}_2(\Omega) \to \mathbb{R}$, $g : \mathbb{L}_2(\Omega) \to \mathbb{R}$, c and $b : J \to \mathbb{R}$, with, $J = [0, T]$.
The novelty of the above problem is in:
-Introducing sequential mean square derivatives in random differential equations,
-Considering the mean square derivative in the right hand sides of the problem,
-Also, the above problem is nonlinear and it is more general than the two problems considered in [11, 18] that are cited above.
The paper is organized as follows: in the next section, we recall all the necessary definitions and lemmas used in the rest of our work. Then, we prove a result on the random integral solution of the problem. After that, we obtain an existence and uniqueness "sequential" result in an appropriate Banach space. In the last section, we introduce new notions and we establish other results of random/detreminisitc continuous and differentially dependence.

1 Preliminaries

In this section, we present some definitions and notations of fractional calculus, and some basic mean square results that we need it in this work [9, 10, 12, 13].

Let $(\Omega, E, \mathbb{P})$ be a compete probability space. Let $X(t, \omega) = \{X(t), t \in J = [0, T], \omega \in \Omega\}$, be a second-order random variable, i.e., $E(X^2(t)) := \int_\Omega X^2 d\mathbb{P} < \infty$. Let $\mathbb{L}_2(\Omega)$ is the Banach space of random variables $X(t) : \Omega \to \mathbb{R}$ such that $E(X^2) < \infty$.
Let $\mathcal{C} = \mathcal{C}(J, \mathbb{L}_2(\Omega))$ the Banach space of the class of all mean continuous second order random processes with the norm

$$
||X||_\mathcal{C} = \sup_{t \in J} ||X||_2 = \sup_{t \in J} \sqrt{E(X(t))^2}.
$$

Now, we have the following definitions.

Definition 1.1. *Let $X(t) \in \mathcal{C}$ and $p > 0$. The mean square Riemann-Liouville fractional integral of order p of $X(t)$ is defined as*

$$\mathbf{I}^p X(t) := \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} X(s) ds,$$

where, $\Gamma(\cdot)$ denotes the gamma function.

Definition 1.2. *Let $X(t) \in \mathcal{C}$ and $q > 0$. The mean square Caputo derivative of fractional order q is defined as:*

$$\mathbf{D}^q X(t) := \frac{1}{\Gamma(n - q)} \int_0^t (t - s)^{n-q-1} X^{(n)}(s) ds, n - 1 < q < n, \ n = [q] + 1, where, n \in \mathbb{N}^*,$$

where, $X^{(n)}$ denotes the mean square differentiation and $X(t)$ is assumed to be mean square differentiable.

Lemma 1.3. *Let $X(t) \in \mathcal{C}$. For $q > 0$, the general solution of the differential equation $\mathbf{D}^q X(t) = 0$, is given by*

$$X(t) = C_0 + C_1 t - + \cdots + C_{n-1} t^{n-1},$$

where, $C_i \in \mathbb{R}, i = 1, \ldots, n - 1, n = [q] + 1$.

Lemma 1.4. *Let $X(t) \in \mathcal{C}$. Let $q > 0$, so,*

$$\mathbf{I}^q \mathbf{D}^q X(t) = X(t) + C_0 + C_1 t + \cdots + C_{n-1} t^{n-1},$$

where, $C_i \in \mathbb{R}, i = 1, \ldots, n - 1, n = [q] + 1$.

2 A Sequential Random Integral Solution

We begin this section by proving the following lemma.

Lemma 2.1. *The integral solution of the sequential random FDE (1) is given by the following formula*

$$\begin{aligned}
X(t) = & a^{-1} \left[X_0 - X_1 \sum_{k=1}^n a_k \tau_k - \sum_{k=1}^n a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)} \left[c(s) f(X(s)) + b(s) g(\mathbf{D}^{\alpha-1} X(s)) \right] ds \right] \\
& + X_1 t + \int_0^t \frac{(t - s)^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)} \left[c(s) f(X(s)) + b(s) g(\mathbf{D}^{\alpha-1} X(s)) \right] ds,
\end{aligned} \tag{2}$$

where, $a = 1 + \sum_{k=1}^n a_k$.

Proof. We note

$$Y(t) := c(t) f(X(t)) + b(t) g(\mathbf{D}^\beta X(t))$$

and we consider

$$\mathbf{D}^\alpha (\mathbf{D}^\beta X)(t) = Y(t) \tag{3}$$

for which we apply the mean square Riemann-Liouville fractional integral of order α to (3) to obtain

$$\mathbf{D}^\beta X(t) = \gamma_0 + \mathbf{I}^\alpha Y(t), \tag{4}$$

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where, $\gamma_0 \in \mathbb{R}$. Again, we apply the mean square Riemann-Liouville fractional integral of order β to (4). We can write

$$X(t) = \gamma_1 + \gamma_0 t + \mathbf{I}^{\alpha+\beta} Y(t), \tag{5}$$

where, $\gamma_1 \in \mathbb{R}$. We take $t = 0$ in (5), we get $X(0) = \gamma_1$, and we take $t = \tau_k$ in (5), we get,

$$X(\tau_k) = \gamma_1 + \gamma_0 \tau_k + \mathbf{I}^{\alpha+\beta} Y(t)\big|_{t=\tau_k},$$

so,

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = \gamma_1 + \sum_{k=1}^{n} a_k \left[\gamma_1 + \gamma_0 \tau_k + \mathbf{I}^{\alpha+\beta} Y(t)\big|_{t=\tau_k} \right]. \tag{6}$$

The derivative of (5) is

$$X'(t) = \gamma_0 + \mathbf{I}^{\alpha+\beta-1} Y(t),$$

and we take $t = 0$, we have

$$X'(0) = \gamma_0 = X_1.$$

Substituting the value of γ_0 in (6), we get the value of γ_1

$$\gamma_1 = \frac{1}{1 + \sum_{k=1}^{n} a_k} \left[X_0 - X_1 \sum_{k=1}^{n} a_k \tau_k - \sum_{k=1}^{n} a_k \, \mathbf{I}^{\alpha+\beta} Y(t)\big|_{t=\tau_k} \right].$$

The proof is thus achieved.. $\qquad\square$

3   A Unique Sequential Solution in the Sense of $\|.\|_F$

Now, we introduce the Banach space

$$F := \{ X \in \mathcal{C}, \mathbf{D}^{\beta} X \in \mathcal{C} \},$$

equipped with the norm

$$\|X\|_F = \|X\|_{\mathcal{C}} + \|\mathbf{D}^{\beta} X\|_{\mathcal{C}}.$$

We define the operator over the space F as follows

$$\begin{aligned} \Phi \; &: F \;\to\; F \\ &\;\; X \;\to\; \Phi X, \end{aligned} \tag{7}$$

$$\Phi X(t) := a^{-1} \left[X_0 - X_1 \sum_{k=1}^{n} a_k \tau_k - \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left[c(s) f(X(s)) + b(s) g\big(\mathbf{D}^{\alpha-1} X(s)\big) \right] ds \right] + $$
$$X_1 t + \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left[c(s) f(X(s)) + b(s) g\big(\mathbf{D}^{\alpha-1} X(s)\big) \right] ds,$$

where, $a = 1 + \sum_{k=1}^{n} a_k$. We prove the following result.

Lemma 3.1. *Suppose that $f, g : \mathbb{L}_2(\Omega) \to \mathbb{R}$ and, $c, b : J \to \mathbb{R}$ are continuous functions. In addition, we assume that*

(H1): $\exists k_1, k_2 > 0, \forall x, y \in \mathbb{L}_2(\Omega)$

$$||f(x) - f(y)||_2 \le K_1 ||x - y||_2,$$

and,

$$||g(x) - g(y)||_2 \le K_2 ||x - y||_2$$

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(H.2): $\sup\limits_{t\in J} |c(t)| = u < \infty$, *and,* $\sup\limits_{t\in J} |b(t)| = v < \infty$.

Then, we have $\Phi : F \to F$.

Proof. Before starting the proof, we define $\sup\limits_{t\in J} f(0) = m_1 < \infty$, *and,* $\sup\limits_{t\in J} g(0) = m_2 < \infty$.

Let $X \in F$, and $\forall t_1, t_2 \in J$, where, $|t_2 - t_1| \le \delta$, we have

$$
\begin{aligned}
\Phi X(t_2) - \Phi X(t_1) =& X_1 t_2 + \int_0^{t_2} \frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big) \right] ds \\
&- X_1 t_1 - \int_0^{t_1} \frac{(t_1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left[c(s)f\big(X(t)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big) \right] ds, \\
=& (t_2-t_1)X_1 + \int_0^{t_1} \frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big) \right] ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big) \right] ds \\
&- \int_0^{t_1} \frac{(t_1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left[c(s)f\big(X(t)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big) \right] ds, \\
=& (t_2-t_1)X_1 + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big) \right] ds \\
&+ \int_0^{t_1} \left[\frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} - \frac{(t_1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \right] \left[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big) \right] ds,
\end{aligned}
$$

Using the above introduced norm

$$
\begin{aligned}
||\Phi X(t_2) - \Phi X(t_1)||_2 \le & |(t_2-t_1)|X_1 + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left[|c(s)|||f\big(X(s)\big)||_2 + |b(s)|||g\big(\mathbf{D}^\beta X(s)\big)||_2 \right] ds + \\
& \int_0^{t_1} \left[\frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} - \frac{(t_1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \right] \left[|c(s)|||f\big(X(s)\big)||_2 + |b(s)|||g\big(\mathbf{D}^\beta X(s)\big)||_2 \right] ds.
\end{aligned}
$$
(8)

We have $|t_2 - t_1| \le \delta$, and

$$
||f\big(X(t)\big)||_2 - |f(0)| \le ||f\big(X(t)\big) - f(0)||_2,
$$

and, according to the assumptions $(H.1)$, we obtain,

$$
\sup_{t\in[0,T]} ||f\big(X(t)\big)||_2 \le K_1||X(t)||_2 + m_1.
$$

With the same arguments for g and according to $(H.1)$, we get

$$
\sup_{t\in[0,T]} ||g\big(\mathbf{D}^\beta X(t)\big)||_2 \le K_2||X(t)||_2 + m_2.
$$

Therefore, we have

$$
\begin{aligned}
||\Phi X(t_2) - \Phi X(t_1)||_{\mathcal{C}} \le & \delta X_1 + \frac{(t_2-t_1)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \left[u\big(K_1||X(t)||_{\mathcal{C}} + m_1\big) + v\big(K_2||X(t)||_2 + m_{\mathcal{C}}\big) \right] \\
& + \left[\frac{-(t_2-t_1)^{\alpha+\beta} + t_2^{\alpha+\beta} - t_1^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right] \left[u\big(K_1||X(t)||_{\mathcal{C}} + m_1\big) + v\big(K_2||X(t)||_{\mathcal{C}} + m_2\big) \right], \\
\le & \delta X_1 + \left[\frac{t_2^{\alpha+\beta} - t_1^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right] \left[u\big(K_1||X(t)||_{\mathcal{C}} + m_1\big) + v\big(K_2||X(t)||_{\mathcal{C}} + m_2\big) \right],
\end{aligned}
$$
(9)

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On the other hand, we have

$$\mathbf{D}^\beta\Phi X(t_2) - \mathbf{D}^\beta\Phi X(t_1) = X_1\frac{t_2^{1-\beta}}{\Gamma(2-\beta)} + \int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\left[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big)\right]ds$$
$$- X_1\frac{t_1^{1-\beta}}{\Gamma(2-\beta)} - \int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big)\right]ds,$$

(10)

Hence,

$$||\mathbf{D}^\beta\Phi X(t_2) - \mathbf{D}^\beta\Phi X(t_1)||_2 \leq \int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\left[|c(s)|||f\big(X(s)\big)||_2 + |b(s)|||g\big(\mathbf{D}^\beta X(s)\big)||_2\right]ds$$
$$- \int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[|c(s)|||f\big(X(s)\big)||_2 + |b(s)|||g\big(\mathbf{D}^\beta X(s)\big)||_2\right]ds$$
$$+ X_1\frac{t_2^{1-\beta} - t_1^{1-\beta}}{\Gamma(2-\beta)},$$

(11)

we pass to the sup of $||\cdot||_2$ on the interval J, we obtain

$$||\mathbf{D}^\beta\Phi X(t_2) - \mathbf{D}^\beta\Phi X(t_1)||_{\mathcal{C}} \leq X_1\frac{t_2^{1-\beta} - t_1^{1-\beta}}{\Gamma(2-\beta)} + \left[\frac{t_2^\alpha - t_1^\alpha}{\Gamma(\alpha+1)}\right]\left[u\big(K_1||X(t)||_{\mathcal{C}} + m_1\big) + v\big(K_2||X(t)||_{\mathcal{C}} + m_2\big)\right].$$

(12)

From inequalities (9) and (12), it yields that

$$||\mathbf{D}^\beta\Phi X(t_2) - \mathbf{D}^\beta\Phi X(t_1)||_F \leq ||\Phi X(t_2) - \Phi X(t_1)||_{\mathcal{C}} + ||\mathbf{D}^\beta\Phi X(t_2) - \mathbf{D}^\beta\Phi X(t_1)||_{\mathcal{C}},$$
$$\leq \left[\delta + \frac{t_2^{1-\beta} - t_1^{1-\beta}}{\Gamma(2-\beta)}\right]X_1 + \left[\frac{t_2^{\alpha+\beta} - t_1^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{t_2^\alpha - t_1^\alpha}{\Gamma(\alpha+1)}\right]$$
$$\times \left[u\big(K_1||X(t)||_{\mathcal{C}} + m_1\big) + v\big(K_2||X(t)||_{\mathcal{C}} + m_2\big)\right] \to 0, \text{ as, } t_2 \to t_1.$$

(13)

Hence, the lemma is proved. □

Using the above Banach space with its introduced norm, we shall prove the existence and uniqueness of sequential solutions of the random FDE.

Theorem 3.2. *Assume that (H.1) and (H.2) hold. The sequential random problem (1) has a unique solution provided that $A < 1$, where,*

$$A := \left(2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)}\right)\left(uK_1 + vK_2\right).$$

Proof. Let $X, Y \in F$, we have

$$\Phi X(t) - \Phi Y(t) = - a^{-1}\sum_{k=1}^n a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big)\right]ds$$
$$+ \int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big)\right]ds$$
$$+ a^{-1}\sum_{k=1}^n a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f\big(Y(s)\big) + b(s)g\big(\mathbf{D}^\beta Y(s)\big)\right]ds$$
$$- \int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f\big(Y(s)\big) + b(s)g\big(\mathbf{D}^\beta Y(s)\big)\right]ds,$$

(14)

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and this implies that

$$\Phi X(t) - \Phi Y(t) =$$

$$- a^{-1} \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left[c(s)\big(f\big(X(s)\big) - f\big(Y(s)\big)\big) + b(s)\big(g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta Y(s)\big)\big) \right] ds$$

$$+ \int_0^t \frac{(t - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left[c(s)\big(f\big(X(s)\big) - f\big(Y(s)\big)\big) + b(s)\big(g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta Y(s)\big)\big) \right] ds, \tag{15}$$

and consequently,

$$||\Phi X(t) - \Phi Y(t)||_2 \leq$$

$$a^{-1} \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left[|c(s)|||f\big(X(s)\big) - f\big(Y(s)\big)||_2 + |b(s)|||(g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta Y(s)\big)||_2 \right] ds$$

$$+ \int_0^t \frac{(t - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left[|c(s)|||f\big(X(s)\big) - f\big(Y(s)\big)||_2 + |b(s)|||(g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta Y(s)\big)||_2 \right] ds. \tag{16}$$

Therefore,

$$||\Phi X(t) - \Phi Y(t)||_\mathcal{C} \leq a^{-1} \sum_{k=1}^{n} a_k \frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \left[uK_1||X - Y||_\mathcal{C} + vK_2||X - Y||_\mathcal{C} \right]$$

$$+ \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \left[uK_1||X - Y||_\mathcal{C} + vK_2||X - Y||_\mathcal{C} \right], \tag{17}$$

thus,

$$||\Phi X(t) - \Phi Y(t)||_\mathcal{C} \leq 2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \left[uK_1||X - Y||_\mathcal{C} + vK_2||X - Y||_\mathcal{C} \right]. \tag{18}$$

Moreover, we have

$$\Phi \mathbf{D}^\beta X(t) - \Phi \mathbf{D}^\beta Y(t) = \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \left[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big) \right] ds$$

$$- \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \left[c(s)f\big(Y(s)\big) + b(s)g\big(\mathbf{D}^\beta Y(s)\big) \right] ds,$$

$$= \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \left[c(s)\big(f\big(X(s)\big) - f\big(Y(s)\big)\big) + b(s)\big(g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta Y(s)\big)\big) \right] ds. \tag{19}$$

Hence, it yields that

$$||\Phi \mathbf{D}^\beta X(t) - \Phi \mathbf{D}^\beta Y(t)||_2 \leq \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \left[|c(s)|||f\big(X(s)\big) - f\big(Y(s)\big)||_2 \right.$$

$$\left. + |b(s)|||(g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta Y(s)\big)||_2 \right] ds, \tag{20}$$

so,

$$||\Phi \mathbf{D}^\beta X(t) - \Phi \mathbf{D}^\beta Y(t)||_\mathcal{C} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left[uK_1||X - Y||_\mathcal{C} + vK_2||X - Y||_\mathcal{C} \right]. \tag{21}$$

By the inequalities (18) and (21), we get

$$||\Phi X(t) - \Phi Y(t)||_F \leq \left[2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)} \right] \left[uK_1 + vK_2 \right] ||X - Y||_\mathcal{C}, \tag{22}$$

At the end, we conclude that

$$||\Phi X(t) - \Phi Y(t)||_F \leq A||X - Y||_F. \tag{23}$$

Finally, the operator Φ is contractive as $A < 1$. This ends the proof. $\qquad\square$

4 Random and Deterministic Data Dependence

In this section, we establish new concepts for the above sequential random FDE with its nonlocal condition; in addition, we prove the results for the continuous and differentilly dependence on random/deterministic data.

So let us consider the following sequential random FDE with the nonlocal conditions:

$$
\begin{cases}
\mathbf{D}^\alpha\big(\mathbf{D}^\beta X\big)(t) &= c(t)f\big(X(t)\big) + b(t)g\big(\mathbf{D}^\beta X(t)\big), \\
\tilde{X}_0 &= X(0) + \sum_{k=1}^{n} a_k X(\tau_k), \\
\tilde{X}_1 &= X'(0),
\end{cases}
\tag{24}
$$

and, we study the continuous dependance on the random data X_0 and X_1 of the solution of the sequential random problem (1).

Definition 4.1. *The solution $X \in F$ of the sequential random problem (1) is continuously and β-differentially dependent on the random data X_0 and X_1, if for all $\epsilon > 0$, $\exists \delta_0 > 0, \delta_1 > 0$ such that $||X_0 - \tilde{X}_0||_2 \leq \delta_0$, and, $||X_1 - \tilde{X}_1||_2 \leq \delta_1$, $\Rightarrow ||X_0 - \tilde{X}_0||_F \leq \epsilon$.*

Theorem 4.2. *Assume that (H.1) and (H.2) hold. Then, the solution of the sequential random FDE is continuously and β-differentially dependent on X_0 and X_1.*

Proof. Let $X(t)$ as defined in (2) be the solution of the problem (1) and

$$
\begin{aligned}
\tilde{X}(t) =& a^{-1}\left[\tilde{X}_0 - \tilde{X}_1 \sum_{k=1}^{n} a_k \tau_k - \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^\beta \tilde{X}(s)\big)\right]ds\right] + \tilde{X}_1 t \\
&+ \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^\beta \tilde{X}(s)\big)\right]ds,
\end{aligned}
\tag{25}
$$

be the solution of the problem (24). Then,

$$
\begin{aligned}
X(t) - \tilde{X}(t) =& a^{-1}\big(X_0 - \tilde{X}_0\big) - a^{-1}\sum_{k=1}^{n} a_k \tau_k \big(X_1 - \tilde{X}_1\big) - a^{-1}\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
&\times \left[c(s)\big(f\big(X(s)\big) - f\big(\tilde{X}(s)\big)\big) + b(s)\big(g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta \tilde{X}(s)\big)\big)\right]ds + t\big(X_1 - \tilde{X}_1\big) \\
&+ \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)\big(f\big(X(s)\big) - f\big(\tilde{X}(s)\big)\big) + b(s)\big(g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta \tilde{X}(s)\big)\big)\right]ds,
\end{aligned}
\tag{26}
$$

we pass to $||\cdot||_2$ on J, we get

$$
\begin{aligned}
||X(t) - \tilde{X}(t)||_2 \leq& a^{-1}||X_0 - \tilde{X}_0||_2 + a^{-1}\sum_{k=1}^{n} a_k \tau_k ||X_1 - \tilde{X}_1||_2 + a^{-1}\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
&\times \left[|c(s)|||f\big(X(s)\big) - f\big(\tilde{X}(s)\big)||_2 + |b(s)|||g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta \tilde{X}(s)\big)||_2\right]ds \\
&+ t||X_1 - \tilde{X}_1||_2 + \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[|c(s)|||f\big(X(s)\big) - f\big(\tilde{X}(s)\big)||_2\right. \\
&\left.+ |b(s)|||g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta \tilde{X}(s)\big)||_2\right]ds,
\end{aligned}
\tag{27}
$$

hence,

$$||X(t) - \tilde{X}(t)||_{\mathcal{C}} \leq a^{-1}\delta_0 + a^{-1}\sum_{k=1}^{n} a_k\tau_k\delta_1 + t\delta_1$$

$$+ a^{-1}\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[uK_1||X(s) - \tilde{X}(s)||_{\mathcal{C}} + vK_2||X(s) - \tilde{X}(s)||_{\mathcal{C}}\right]ds$$

$$+ \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[uK_1||X(s) - \tilde{X}(s)||_{\mathcal{C}} + vK_2||X(s) - \tilde{X}(s)||_{\mathcal{C}}\right]ds,$$

$$\leq a^{-1}\delta_0 + a^{-1}\sum_{k=1}^{n} a_k T\delta_1 + T\delta_1$$

$$+ a^{-1}\sum_{k=1}^{n} a_k \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left[uK_1||X(s) - \tilde{X}(s)||_{\mathcal{C}} + vK_2||X(s) - \tilde{X}(s)||_{\mathcal{C}}\right]$$

$$+ \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left[uK_1||X(s) - \tilde{X}(s)||_{\mathcal{C}} + vK_2||X(s) - \tilde{X}(s)||_{\mathcal{C}}\right]. \tag{28}$$

So,

$$||X(t) - \tilde{X}(t)||_{\mathcal{C}} \leq a^{-1}\delta_0 + \left(a^{-1}\sum_{k=1}^{n} a_k\tau_k + 1\right)T\delta_1$$

$$+ \left(a^{-1}\sum_{k=1}^{n} a_k\tau_k + 1\right)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(uK_1 + vK_2\right)||X(s) - \tilde{X}(s)||_{\mathcal{C}},$$

$$||X(t) - \tilde{X}(t)||_{\mathcal{C}} \leq \delta_0 + 2T\delta_1 + 2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(uK_1 + vK_2\right)||X(s) - \tilde{X}(s)||_{\mathcal{C}}. \tag{29}$$

Furthermore,

$$\mathbf{D}^\beta X(t) - \mathbf{D}^\beta \tilde{X}(t) = X_1\frac{t^{1-\beta}}{\Gamma(2-\beta)} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big)\right]ds$$

$$- \tilde{X}_1\frac{t^{1-\beta}}{\Gamma(2-\beta)} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^\beta \tilde{X}(s)\big)\right]ds. \tag{30}$$

Then, we have

$$||\mathbf{D}^\beta X(t) - \mathbf{D}^\beta \tilde{X}(t)||_{\mathcal{C}} \leq ||X_1 - \tilde{X}_1||_2\frac{t^{1-\beta}}{\Gamma(2-\beta)} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[|c(s)|||f\big(X(s)\big) - f\big(\tilde{X}(s)\big)||_2\right.$$

$$\left. + |b(s)|||g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta \tilde{X}(s)\big)||_2\right]ds,$$

$$\leq \delta_1\frac{T^{1-\beta}}{\Gamma(2-\beta)} + \frac{T^\alpha}{\Gamma(\alpha+1)}\left(uK_1 + vK_2\right)||X - \tilde{X}||_{\mathcal{C}}. \tag{31}$$

Combining the inequalities (29) and (31), we obtain

$$||X(t) - \tilde{X}(t)||_F \leq \delta_0 + \left(2T + \frac{T^{1-\beta}}{\Gamma(2-\beta)}\right)\delta_1$$

$$+ \left(2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)}\right)\left(uK_1 + vK_2\right)||X(s) - \tilde{X}(s)||_F \tag{32}$$

witch implies that

$$||X(t) - \tilde{X}(t)||_F \leq \frac{\delta_0 + \left(2T + \frac{T^{1-\beta}}{\Gamma(2-\beta)}\right)\delta_1}{1 - A} = \epsilon. \tag{33}$$

This ends the proof. $\qquad\square$

We pass to study the dependance on the deterministic data $a_k > 0$ of the solution of the sequential random problem (1).
We consider the sequential random FDE with the nonlocal conditions

$$\begin{cases} \mathbf{D}^\alpha(\mathbf{D}^\beta X)(t) &= c(t)f(X(t)) + b(t)g(\mathbf{D}^\beta X(t)), \\ X_0 &= X(0) + \sum_{k=1}^n \tilde{a}_k X(\tau_k), \\ X_1 &= X'(0), \end{cases} \tag{34}$$

and we introduce the following definition.

Definition 4.3. *The solution $X \in F$ of the sequential random FDEs (1) is continuously and β-differentially dependend on the deterministic data a_k if for all $\epsilon > 0, \exists \delta > 0$ such that $|a_k - \tilde{a}_k| < \delta \Rightarrow ||X - \tilde{X}||_F \leq \epsilon$.*

No, we present to the reader the following result:

Theorem 4.4. *Assume that $(H.1)$ and $(H.2)$ hold. Then, the solution of the sequential random FDE is continuously and β-differentially dependent on a_k.*

Proof. Before starting the proof, we introduce the following notations

$$\begin{aligned} \mathcal{K}_1 &= a^{-1} - \tilde{a}^{-1}, \\ \mathcal{K}_2 &= \tilde{a}^{-1}\sum_{k=1}^n \tilde{a}_k - a^{-1}\sum_{k=1}^n a_k, \\ \mathcal{K}_3 &= \tilde{a}^{-1}\sum_{k=1}^n \tilde{a}_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f(\tilde{X}(s)) + b(s)g(\mathbf{D}^\beta \tilde{X}(s))\right]ds \\ &\quad - a^{-1}\sum_{k=1}^n a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f(X(s)) + b(s)g(\mathbf{D}^\beta X(s))\right]ds, \\ \mathcal{K}_4 &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f(X(s)) + b(s)g(\mathbf{D}^\beta X(s))\right]ds \\ &\quad - \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f(\tilde{X}(s)) + b(s)g(\mathbf{D}^\beta \tilde{X}(s))\right]ds. \end{aligned}$$

Let $X(t)$ as defined in equation (2) be the solution of the problem (1) and

$$\begin{aligned} \tilde{X}(t) = \tilde{a}^{-1}&\left[X_0 - X_1 \sum_{k=1}^n \tilde{a}_k\tau_k - \sum_{k=1}^n \tilde{a}_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f(\tilde{X}(s)) + b(s)g(\mathbf{D}^\beta \tilde{X}(s))\right]ds\right] + X_1 t \\ &+ \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f(\tilde{X}(s)) + b(s)g(\mathbf{D}^\beta \tilde{X}(s))\right]ds, \end{aligned} \tag{35}$$

be the solution of the problem (34). Then,

$$X(t) - \tilde{X}(t) = \mathcal{K}_1 X_0 + \tau_k \mathcal{K}_2 X_1 + \mathcal{K}_3 + \mathcal{K}_4. \tag{36}$$

Hence, we get

$$|\mathcal{K}_1| \leq |\sum_{k=1}^{n} \tilde{a}_k - \sum_{k=1}^{n} a_k| \tag{37}$$
$$\leq n\delta,$$

and

$$\mathcal{K}_3 = \tilde{a}^{-1}\Big(1 + \sum_{k=1}^{n} \tilde{a}_k\Big) \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^\beta \tilde{X}(s)\big)\Big] ds$$
$$- a^{-1}\Big(1 + \sum_{k=1}^{n} a_k\Big) \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big)\Big] ds$$
$$- \tilde{a}^{-1} \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^\beta \tilde{X}(s)\big)\Big] ds$$
$$+ a^{-1} \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big)\Big] ds, \tag{38}$$

so,

$$\mathcal{K}_3 = - \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)(f\big(X(s)\big) - f\big(\tilde{X}(s)\big)) + b(s)(g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta \tilde{X}(s)\big))\Big] ds$$
$$+ (a^{-1} - \tilde{a}^{-1}) \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big)\Big] ds$$
$$+ \tilde{a}^{-1} \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)(f\big(X(s)\big) - f\big(\tilde{X}(s)\big)) + b(s)(g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta \tilde{X}(s)\big))\Big] ds, \tag{39}$$

we know that
$$\sup_{t\in[0,T]} ||f\big(X(t)\big)||_2 \leq K_1 ||X(t)||_2 + m_1,$$
and,
$$\sup_{t\in[0,T]} ||g\big(\mathbf{D}^\beta X(t)\big)||_2 \leq K_2 ||X(t)||_2 + m_2.$$
By using our hypotheses, we get

$$||\mathcal{K}_3||_2 \leq \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[uK_1||X(s) - \tilde{X}(s)||_2 + vK_2||X(s) - \tilde{X}(s)||_2\Big] ds$$
$$+ n\delta \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[u(K_1||X(s)||_2 + m_1) + v(K_2||X(s)||_2 + m_2)\Big] ds \tag{40}$$
$$+ \tilde{a}^{-1} \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[uK_1||X(s) - \tilde{X}(s)||_2 + vK_2||X(s) - \tilde{X}(s)||_2\Big] ds,$$

hence,

$$||\mathcal{K}_3||_2 \leq \frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Big[uK_1||X(s) - \tilde{X}(s)||_2 + vK_2||X(s) - \tilde{X}(s)||_2\Big]$$
$$+ n\delta \frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Big[u(K_1||X(s)||_2 + m_1) + v(K_2||X(s)||_2 + m_2)\Big] \tag{41}$$
$$+ \tilde{a}^{-1} \frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Big[uK_1||X(s) - \tilde{X}(s)||_2 + vK_2||X(s) - \tilde{X}(s)||_2\Big].$$

By $(H.1)$ and $(H.2)$,

$$||\mathcal{K}_4||_2 \leq \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left[uK_1||X(s)-\tilde{X}(s)||_2 + vK_2||X(s)-\tilde{X}(s)||_2\right]. \tag{42}$$

Then,

$$||X(t)-\tilde{X}(t)||_2 \leq |\mathcal{K}_1|||X_0||_2 + \tau_k||\mathcal{K}_2||_2||X_1||_2 + ||\mathcal{K}_3||_2 + ||\mathcal{K}_4||_2,$$
$$\leq n\delta||X_0||_2 + \tau_k n\delta||X_1||_2 + \left(1+\tilde{a}^{-1}\right)\frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(uK_1+vK_2\right)||X(s)-\tilde{X}(s)||_2$$
$$+ n\delta\frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left[u(K_1||X(s)||_2+m_1)+v(K_2||X(s)||_2+m_2)\right]$$
$$+ \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(uK_1+vK_2\right)||X(s)-\tilde{X}(s)||_2, \tag{43}$$

we pass now to the sup over J, t yields that

$$||X-\tilde{X}||_{\mathcal{C}} \leq n\delta\left[||X_0||_2 + T||X_1||_2 + \frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(u(K_1||X||_{\mathcal{C}}+m_1)+v(K_2||X||_{\mathcal{C}}+m_2)\right)\right]$$
$$+ 3\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(uK_1+vK_2\right)||X-\tilde{X}||_{\mathcal{C}}. \tag{44}$$

Also, we have

$$\mathbf{D}^\beta X(t) - \mathbf{D}^\beta \tilde{X}(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[c(s)(f(X(s))-f(\tilde{X}(s)))+b(s)(g(\mathbf{D}^\beta X(s))-g(\mathbf{D}^\beta \tilde{X}(s)))\right]ds \tag{45}$$

so,

$$||\mathbf{D}^\beta X - \mathbf{D}^\beta \tilde{X}||_{\mathcal{C}} \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\left(uK_1+vK_2\right)||X-\tilde{X}||_{\mathcal{C}}. \tag{46}$$

By the inequalities (44) and (46), we observe that

$$||X-\tilde{X}||_F \leq n\delta\left[||X_0||_2 + T||X_1||_2 + \frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(u(K_1||X||_{\mathcal{C}}+m_1)+v(K_2||X||_{\mathcal{C}}+m_2)\right)\right]$$
$$+ \left[\frac{T^\alpha}{\Gamma(\alpha+1)} + 3\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right]\left[uK_1+vK_2\right]||X-\tilde{X}||_{\mathcal{C}},$$
$$\leq \frac{n\delta\left[||X_0||_2 + T||X_1||_2 + \frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(u(K_1||X||_{\mathcal{C}}+m_1)+v(K_2||X||_{\mathcal{C}}+m_2)\right)\right]}{1-L}, \tag{47}$$

where, $L = \left[\frac{T^\alpha}{\Gamma(\alpha+1)} + 3\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right]\left[uK_1+vK_2\right]$. $\qquad\square$

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