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Some Orthogonal Polynomials on the Finite Interval and Gaussian Quadrature Rules for Fractional Riemann-Liouville Integrals

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Inspired by papers by M. A. Bokhari, A. Qadir, and H. Al-Attas [On Gauss-type quadrature rules, *Numer. Funct. Anal. Optim.* **31 (2010), 1120–1134] and by M. R. Rapaić, T. B. Šekara, and V. Govedarica [A novel class of fractionally orthogonal quasi-polynomials and new fractional quadrature formulas,** *Appl. Math. Comput.* **245 (2014), 206–219], in this paper we investigate a few types of orthogonal polynomials on finite intervals and derive the corresponding quadrature formulas of Gaussian type for efficient numerical computation of the left and right fractional Riemann-Liouville integrals. Several numerical examples are included to demonstrate the numerical efficiency of the proposed procedure. Copyright © 2020 John Wiley & Sons, Ltd.**

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1. Introduction and Preliminaries

Let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree at most n, and $\mathcal{P}$ be the set of all algebraic polynomials. The set of all monic polynomials of degree n will be denoted by $\widehat{\mathcal{P}}_n$, i.e.,

$$\hat{\mathcal{P}}_n = \Big\{ t^n + q(t) \;\big|\; q(t) \in \mathcal{P}_{n-1} \Big\} \subset \mathcal{P}_n.$$

This paper is devoted to certain classes of orthogonal polynomials on the finite interval on the real line, as well as the corresponding quadrature formulas of maximal degree of precision.

The most known orthogonal polynomials are ones on the real line with respect to the inner product defined by

$$\langle p, q \rangle_w = \int_a^b p(t)q(t)w(t)\,\mathrm{d}t \quad (p, q \in \mathcal{P}), \tag{1.1}$$

where $t \mapsto w(t)$ is a non-negative function on (a, b), $-\infty \le a < b \le +\infty$, for which all moments $\mu_k = \int_a^b t^k w(t)\,\mathrm{d}t$, $k = 0, 1, \ldots$, exist and $\mu_0 > 0$. Such a function is known as the *weight function* on (a, b). The inner product (1.1) gives rise to a unique system of *monic orthogonal polynomials* $\pi_n(\,\cdot\,) = \pi_n(\,\cdot\,; w)$, $n \in \mathbb{N}$, which satisfy the three-term recurrence relation

$$\pi_{n+1}(t) = (t - \alpha_n)\pi_n(t) - \beta_n\pi_{n-1}(t), \quad n = 0, 1, \ldots, \tag{1.2}$$

with $\pi_0(t) = 1$ and $\pi_{-1}(t) = 0$, where the recurrence coefficients depend only on the weight function w, i.e., $\alpha_n = \alpha_n(w)$ and $\beta_n = \beta_n(w)$. The coefficients β_k, $k \ge 1$, in (1.2) are positive, and β_0 may be arbitrary, but sometimes it is convenient to define it by $\beta_0 = \mu_0 = \int_a^b w(t)\,\mathrm{d}t$. All zeros of $\pi_n(t)$, $n \in \mathbb{N}$, are real and distinct and are located in the interior of (a, b). Using numerical methods of linear algebra (QR or QL algorithm), it is easy to compute the zeros τ_k, $k = 1, \ldots, n$, of the orthogonal polynomials

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$\pi_n(t)$ rapidly and efficiently as eigenvalues of the so-called *Jacobi matrix* of order n associated with the weight function w,

$$J_n(w) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathbf{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}. \tag{1.3}$$

A simplification of QR algorithm, known as the Golub-Welsch procedure [16], enables an efficient construction of the Gaussian quadrature formula with respect to the same weight function w, i.e.,

$$\int_a^b f(t)w(t)\,\mathrm{d}t = \sum_{k=1}^{n} A_k f(\tau_k) + R_n(f;w), \tag{1.4}$$

which is exact on the set $\mathcal{P}_{2n-1}$ $(R_n(\mathcal{P}_{2n-1};w) = 0)$. Such a quadrature formula exists for each $n \in \mathbb{N}$ and has the maximal algebraic degree of exactness $d_{\max} = 2n-1$. Its nodes τ_k, $k = 1,\ldots,n$, are exactly eigenvalues of the Jacobi matrix $J_n(w)$ given by (1.3), and the weight coefficients A_k, the so-called Christoffel numbers, are given by $A_k = \beta_0 v_{k,1}^2$, $k = 1,\ldots,n$, where $\beta_0 = \mu_0 = \int_a^b w(t)\,\mathrm{d}t$ and $v_{k,1}$ is the first component of the normalized eigenvector $\mathbf{v}_k$ $(= [v_{k,1} \ \ldots \ v_{k,n}]^{\mathrm{T}})$ corresponding to the eigenvalue τ_k,

$$J_n(w)\mathbf{v}_k = \tau_k \mathbf{v}_k, \quad \mathbf{v}_k^{\mathrm{T}}\mathbf{v}_k = 1, \quad k = 1,\ldots,n.$$

Unfortunately, the recursion coefficients α_n and β_n in (1.2) are known explicitly only for some narrow classes of orthogonal polynomials. One of the most important classes for which these coefficients are known explicitly are the so-called *very classical* orthogonal polynomials (Jacobi, generalized Laguerre, and Hermite polynomials). They are very popular because of several interesting common properties and many applications in numerical analysis, approximation theory, differential equations, as well as in physics, chemistry, electrotechnics, and other computational and applied sciences. Classical orthogonal polynomials possess a number of interesting properties (Rodrigues' type formula, characterization by differential equations, etc.). For details see [9], [31], [23], as well as for different characterizations [1, 2, 5].

Orthogonal polynomials for which the recursion coefficients are not known we call *strongly non-classical polynomials*. Important advances in the numerical construction of recursion coefficients for strongly non-classical polynomials were made by Walter Gautschi in the eighties of the last century, developing the so-called *constructive theory of orthogonal polynomials* [12] (see also [13], [14], [24]). The main problem in this construction is very high sensitivity of recursion coefficients with respect to small perturbations in the data for the moments. Therefore, he developed a few methods for construction (e.g., method of modified moments and discretized Stieltjes-Gautschi procedure), gave a detailed stability analysis of such algorithms as well as several new applications of orthogonal polynomials (for details see [24]).

Most recent, advances in symbolic computation and variable precision arithmetic have made it possible to overcome sensitivity problems, directly by using the original Chebyshev method of moments, but we need then a procedure for the symbolic calculation of moments or their calculation in variable-precision arithmetic (in sufficiently high precision). For such purpose we use our Mathematica package `OrthogonalPolynomials` (see [11], [26]). The package is downloadable from Web Site: `http://www.mi.sanu.ac.rs/ gvm/`. The alternative package `SOPQ` in Matlab was developed by Gautschi (cf. [14], [15]).

Inspired by the recent papers [8], [25], [29], in this paper we consider certain classes of orthogonal polynomials on the finite intervals and derive the corresponding quadrature formulas of the maximal algebraic degree of precision, which can be successfully applied in numerical calculation of the left and right fractional Riemann-Liouville integrals (cf. [20], [21], [7], [6])

$${}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau \quad \text{and} \quad {}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \tag{1.5}$$

respectively, as well as their multiple composition, recently introduced in [10].

The paper is organized as follows. In Section 2 we analyze results from [8] and [25], and consider the generalized case with the weight function

$$w(x) = B_\alpha(x) = \begin{cases} 1-|x|^\alpha, & -1 \le x \le 1, \\ 0, & \text{otherwise}. \end{cases} \tag{1.6}$$

In Section 3 we study a more general case with the weight function

$$w(x) = W_{\alpha,\beta}(x) = \begin{cases} |x|^{2/\beta-1}(1-|x|^{2/\beta})^{\alpha-1}, & -1 \le x \le 1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.7}$$

where $\alpha, \beta > 0$. Orthogonality on $(0,1)$, inspired by results from [29], is considered in Section 4, as well as the corresponding weighted quadrature formulas of Gaussian type. This kind of orthogonality is connected to the orthogonality on the symmetric interval $(-1,1)$ with respect to the weight function (1.7). A procedure for numerical computation of of the left and right fractional Riemann-Liouville integrals (1.5), based on weighted Gauss-Christoffel quadrature rules, is proposed in 5. Several numerical examples are also included in order to demonstrate the numerical efficiency of the proposed procedure.

In some parts of this paper we use the generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters defined by (cf. [28])

$$ {}_pF_q\left[\begin{array}{c}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{array}\middle|\, z\right] = {}_pF_q[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z] = \sum_{\nu=0}^{\infty}\frac{(\alpha_1)_\nu\cdots(\alpha_p)_\nu}{(\beta_1)_\nu\cdots(\beta_q)_\nu}\cdot\frac{z^\nu}{\nu!}, \tag{1.8}$$

where $(\alpha)_n$ $(\alpha\in\mathbb{C})$ denotes the Pochhammer symbol, defined by $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n\in\mathbb{N}$ and $(\alpha)_0=1$. Using the fundamental functional relation for Euler's Gamma function, $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, the Pochhammer symbol $(\alpha)_n$ can be written in the form $(\alpha)_n=\Gamma(\alpha+n)/\Gamma(\alpha)$ $(n\in\mathbb{N}_0)$. For convergence condition and properties of ${}_pF_q$, we refer [28].

2. Orthogonal Polynomials and Gaussian Quadrature Rules with Respect to the Weight Function (1.6)

In [8] Bokhari, Qadir, and Al-Attas considered Gauss quadrature rules, as well as Gauss-Radau and Gauss-Lobatto rules, based on polynomials $p_n(t)$ orthogonal on $(0,1)$ with respect to the linear weight function $\omega(t):=1-t$. The authors discussed a development of the corresponding orthogonal polynomials $p_n(t)$ via Gaussian hypergeometric differential equation, narrated some of its properties, derived the three-term recurrence relation for the monic polynomials (see [8, Theorem 2])

$$p_{n+1}(t)=\left(t-\frac{2(n+1)^2-1}{4(n+1)^2-1}\right)p_n(t)-\frac{n(n+1)}{4(2n+1)^2}p_{n-1}(t),\qquad n=0,1,\ldots, \tag{2.1}$$

where $p_0(t)=1$ and $p_{-1}(t)=0$, and considered several numerical examples of such kind of quadratures.

In a short note we show that these polynomials $p_n(t)$ are a special case of the well-known Jacobi polynomials on $(0,1)$ (see [25]). Namely, by a change of variables $x=2t-1$ in the classical (monic) Jacobi polynomials $\widehat{P}_n^{(\alpha,\beta)}$ (cf. [23, pp. 131–140]), we can easily get the monic orthogonal polynomials $p_n^{(\alpha,\beta)}(t)$ $\left(=2^{-n}\widehat{P}_n^{(\alpha,\beta)}(2t-1)\right)$ orthogonal on $(0,1)$, with respect to the weight function $t\mapsto\omega(t):=(1-t)^\alpha t^\beta$, $\alpha,\beta>-1$, as well as their three-term recurrence relation

$$p_{n+1}^{(\alpha,\beta)}(t)=(x-\alpha_n)p_n^{(\alpha,\beta)}(t)-\beta_n p_{n-1}^{(\alpha,\beta)}(t),\qquad n=0,1\ldots, \tag{2.2}$$

with the recursive coefficients

$$\begin{cases}\alpha_n &= \dfrac{(2n+\alpha+\beta+1)^2-(1+\alpha^2-\beta^2)}{2\left[(2n+\alpha+\beta+1)^2-1\right]} \qquad (n\ge 0),\\[2ex] \beta_n &= \dfrac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2\left[(2n+\alpha+\beta)^2-1\right]} \qquad (n\ge 1).\end{cases} \tag{2.3}$$

For $\alpha=1$ and $\beta=0$, the relation (2.2) reduces to (2.1), i.e., the polynomials $p_n(t)$ discussed in [8] can be expressed in terms of transformed Jacobi polynomials

$$p_n(t)=p_n^{(\alpha,\beta)}(t)=2^{-n}\widehat{P}_n^{(\alpha,\beta)}(2t-1)\big).$$

Several other particular cases were listed in [25].

The relation (2.2) can be also obtained taking the even weight function $x\mapsto w(x)=|x|^\gamma(1-x^2)^\alpha$ on $(-1,1)$, with $\gamma,\alpha>-1$, and the corresponding generalized Gegenbauer polynomials $W_n^{(\alpha,\beta)}(x)$, $\beta=(\gamma-1)/2$, which were introduced by Laščenov [22] (see, also, [9, pp. 155–156], [23, pp. 147–148]). Their three-term recurrence relation is

$$W_{n+1}^{(\alpha,\beta)}(x)=xW_n^{(\alpha,\beta)}(x)-B_nW_{n-1}^{(\alpha,\beta)}(x),\qquad n=0,1,\ldots, \tag{2.4}$$

with the starting polynomials $W_0^{(\alpha,\beta)}(x)=1$ and $W_{-1}^{(\alpha,\beta)}(x)=0$, and the recursion coefficients

$$B_{2n}=\frac{n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \tag{2.5}$$

$$B_{2n-1}=\frac{(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)}, \tag{2.6}$$

except $\alpha+\beta=-1$; then $B_1=\beta+1$.

Remark 2.1 It is interesting that the Laščenov polynomials were rediscovered in 1969 in a M. S. Thesis [30], where the author considered the weight function $\mapsto |x|^\alpha(1-x^2)^\beta$ on $[-1,1]$, with $\alpha,\beta>-1$ and obtained a compact expression for the coefficients B_n in the form

$$B_n=\frac{\left(\alpha\sin^2\left(\frac{\pi n}{2}\right)+n\right)\left(2\beta+\alpha\sin^2\left(\frac{\pi n}{2}\right)+n\right)}{(\alpha+2\beta+2n-1)(\alpha+2\beta+2n+1)},\quad n\ge 1.$$

By changing $\alpha:=\gamma=2\beta+1$ and $\beta:=\alpha$, we get the formulas (2.5) and (2.6) in a compact unique form

$$B_n=\frac{1}{4}\cdot\frac{\left[n+(2\beta+1)\sin^2\left(\frac{\pi n}{2}\right)\right]\left[n+2\alpha+(2\beta+1)\sin^2\left(\frac{\pi n}{2}\right)\right]}{(n+\alpha+\beta)(n+\alpha+\beta+1)},\quad n\ge 1.$$

Since the weight function w is even on $(-1,1)$, using Theorems 2.2.11 and 2.2.12 from [23, pp. 102–103], we get (2.2) for polynomials orthogonal with respect to the weight $\omega(t)=w(\sqrt{t})/\sqrt{t}=t^\beta(1-t)^\alpha$, with

$$\alpha_0=B_1=\frac{\beta+1}{\alpha+\beta+2},\quad \alpha_n=B_{2n}+B_{2n+1},\quad \beta_n=B_{2n-1}B_{2n},\quad n\ge 1. \tag{2.7}$$

Now, we consider the weight function $x\mapsto w(x)=B_\alpha(x)$ given by (1.6) for arbitrary $\alpha>0$ and the corresponding Gaussian rules

$$\int_{\mathbb{R}} f(x)B_\alpha(x)\,\mathrm{d}x=\int_{-1}^{1} f(x)(1-|x|^\alpha)\,\mathrm{d}x=\sum_{k=1}^{n}A_k^{(n)}f(x_k^{(n)})+R_n(f;B_\alpha), \tag{2.8}$$

for which $R_n(p;B_\alpha)=0$ for all polynomials p of degree at most $2n-1$.

This weight function $B_\alpha(x):(-1,1)\to\mathbb{R}_+$ is an even extension of $\omega(t)=1-t^\alpha$ from $(0,1)$ to $(-1,1)$. This weight function $B_\alpha(x)$ for several values of α is presented in Fig. 1.

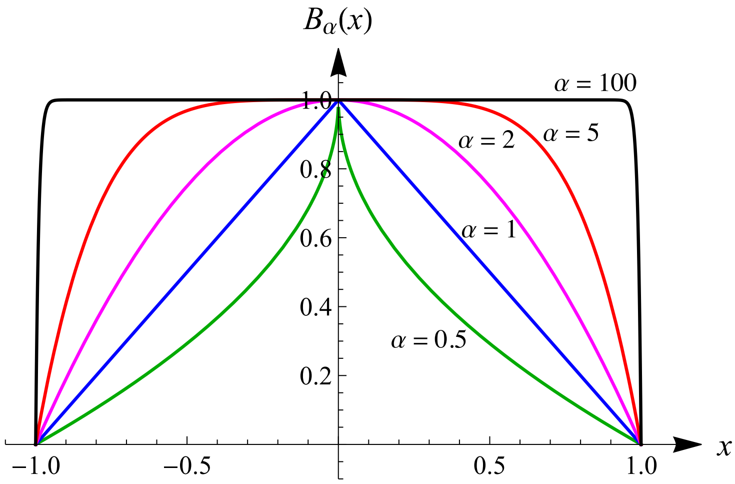

Figure 1. Graphics of the weight function $x\mapsto B_\alpha(x)$ for $\alpha=1/2,1,2,5,$ and 100

In order to construct orthogonal polynomials and the corresponding Gauss-Christoffel quadrature rules up to n nodes in our case we need $2n$ moments

$$\mu_k=\int_{\mathbb{R}} x^k B_\alpha(x)\,\mathrm{d}x=\begin{cases}\dfrac{2\alpha}{(1+k)(1+k+\alpha)}, & k\,(\ge 0)\text{ is even},\\[2mm] 0, & k\,(\ge 1)\text{ is odd}.\end{cases} \tag{2.9}$$

Using our Mathematica Package `OrthogonalPolynomials` (see [11], [26]) and executing the following commands (with $n=50$)

```
<< orthogonalPolynomials`
mom100=Table[If[OddQ[k],0,2a/((1+k)(1+k+a))], {k,0,99}];
{al50,be50}=aChebyshevAlgorithm[mom100,Algorithm -> Symbolic];
```

we obtain the first $n=50$ coefficients in the three-term recurrence relation for the corresponding monic orthogonal polynomials $\pi_k(x)$,

$$\pi_{k+1}(x)=x\pi_k(x)-\beta_k\pi_{k-1}(x),\quad k=1,2,\dots,n-1. \tag{2.10}$$

Note that $\alpha_k = 0$ for each k, because the weight function $x \mapsto B_\alpha(x)$ is even. The obtained coefficients β_k are:

$$\beta_0 = \frac{2\alpha}{\alpha+1}, \quad \beta_1 = \frac{\alpha+1}{3(\alpha+3)}, \quad \beta_2 = \frac{4\left(\alpha^2+6\alpha+14\right)}{15(\alpha+3)(\alpha+5)}, \quad \beta_3 = \frac{9(\alpha+3)^2\left(\alpha^2+10\alpha+46\right)}{35(\alpha+5)(\alpha+7)\left(\alpha^2+6\alpha+14\right)},$$

$$\beta_4 = \frac{16\left(\alpha^6+30\alpha^5+426\alpha^4+3270\alpha^3+14094\alpha^2+33690\alpha+40694\right)}{63(\alpha+7)(\alpha+9)\left(\alpha^2+6\alpha+14\right)\left(\alpha^2+10\alpha+46\right)},$$

$$\beta_5 = \frac{25(\alpha+5)^2\left(\alpha^2+6\alpha+14\right)\left(\alpha^6+42\alpha^5+858\alpha^4+9618\alpha^3+62766\alpha^2+238686\alpha+489254\right)}{99(\alpha+9)(\alpha+11)\left(\alpha^2+10\alpha+46\right)\left(\alpha^6+30\alpha^5+426\alpha^4+3270\alpha^3+14094\alpha^2+33690\alpha+40694\right)},$$

etc.

All computations were performed in Mathematica, Ver. 12.1.0, on MacBook Pro (2017), OS Catalina Ver. 10.15.5. The running time for calculating these symbolic coefficients was about 6 minutes, precisely $6'28''$. If you need less number of coefficients, the running time is drastically shortened. For example, for the first $n = 40$ coefficients this time is $1'53''$, and for $n = 20$ the corresponding running time is only 2 seconds. Otherwise, the running times are evaluated by the function Timing in Mathematica and it includes only CPU time spent in the Mathematica kernel. Because of the use of internal system caches, this can give different results on different occasions within a session. In order to generate worst-case timing results independent of previous computations we used the command ClearSystemCache[].

From the obtained symbolic values of the coefficients β_k we can easily get values for a particular weight function $x \mapsto B_q(x) = 1 - |x|^q$ (for $\alpha = q$), only using a simple command be50/.a->q. In the sequel we give recurrence coefficients for some particular cases:

1° The weight function $\mathbf{B_1(x) = 1 - |x|}$. Here (cf. [25])

$$\beta_0 = 1, \quad \beta_1 = \frac{1}{6}, \quad \beta_2 = \frac{7}{30}, \quad \beta_3 = \frac{57}{245}, \quad \beta_4 = \frac{683}{2793}, \quad \beta_5 = \frac{207725}{856482},$$

$$\beta_6 = \frac{286749501}{1159331030}, \quad \beta_7 = \frac{286268618986}{1164429355245}, \quad \beta_8 = \frac{272609711230510}{1097298927604497},$$

$$\beta_9 = \frac{1098662762497992381 09}{4441688781543149127 74}, \quad \beta_{10} = \frac{1230269378984465608526587}{4941343738726228807816542}, \quad \text{etc.},$$

as well as the corresponding orthogonal polynomials:

$$\pi_0(x) = 1, \quad \pi_1(x) = x, \quad \pi_2(x) = x^2 - \frac{1}{6}, \quad \pi_3(x) = x^3 - \frac{2x}{5},$$

$$\pi_4(x) = x^4 - \frac{31x^2}{49} + \frac{19}{490}, \quad \pi_5(x) = x^5 - \frac{50x^3}{57} + \frac{109x}{798},$$

$$\pi_6(x) = x^6 - \frac{16825x^4}{15026} + \frac{2179x^2}{7513} - \frac{5935}{631092}, \quad \text{etc.}$$

2° The weight function $\mathbf{B_{1/2}(x) = 1 - \sqrt{|x|}}$. Here we have

$$\beta_0 = \frac{2}{3}, \quad \beta_1 = \frac{1}{7}, \quad \beta_2 = \frac{92}{385}, \quad \beta_3 = \frac{287}{1265}, \quad \beta_4 = \frac{13328}{53751}, \quad \beta_5 = \frac{466015}{1946721},$$

$$\beta_6 = \frac{22905388}{91754117}, \quad \beta_7 = \frac{243053089027}{997174601189}, \quad \beta_8 = \frac{370642573889612096}{1481868458865339699},$$

$$\beta_9 = \frac{275010048107533776562 57}{111881203031704489008087}, \quad \beta_{10} = \frac{3645786181918857621770456 9428}{1456708263247610995975288381 87},$$

etc.

3° The weight function $\mathbf{B_2(x) = 1 - x^2}$. Here we obtain

$$\beta_0 = \frac{4}{3}, \quad \beta_1 = \frac{1}{5}, \quad \beta_2 = \frac{8}{35}, \quad \beta_3 = \frac{5}{21}, \quad \beta_4 = \frac{8}{33}, \quad \beta_5 = \frac{35}{143}, \quad \beta_6 = \frac{16}{65}, \quad \text{etc.},$$

i.e.,

$$\beta_0 = \frac{4}{3}, \quad \beta_k = \frac{k(k+2)}{(2k+1)(2k+3)}, \quad k = 1, 2, \ldots,$$

because this is a special case of the Gegenbauer weight $x \mapsto (1-x^2)^{\lambda-1/2}$ for $\lambda = 3/2$. Otherwise, in this general case we have (cf. [13, p. 29])

$$\beta_0 = \sqrt{\pi}\,\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+1)}, \quad \beta_k = \frac{k(k+2\lambda-1)}{4(k+\lambda)(k+\lambda-1)}, \quad k = 1, 2, \ldots . \tag{2.11}$$

Note that by the command `Limit[be50,a->Infinity]` we get

$$\beta_0 = 2, \quad \beta_1 = \frac{1}{3}, \quad \beta_2 = \frac{4}{15}, \quad \beta_3 = \frac{9}{35}, \quad \beta_4 = \frac{16}{63}, \quad \beta_5 = \frac{25}{99}, \quad \beta_6 = \frac{36}{143}, \quad \text{etc.},$$

i.e., the recurrence coefficients for the Legendre weight (a special case of (2.11) for $\lambda = 1/2$)

$$\beta_0 = 2, \quad \beta_k = \frac{k^2}{4k^2-1}, \quad k = 1, 2, \ldots .$$

In order to construct the n-point Gaussian quadrature rule (2.8) for each $n \le N$, we need first N coefficients β_k, $k = 0, 1, \ldots, N-1$, i.e., the sequence beta (alpha is a zero sequence, because $w(x)$ is an even weight function), obtained from the first $2N-1$ moments (2.9). The Gaussian quadrature parameters, the nodes $x_k^{(n)}$ and the weight coefficients $A_k^{(n)}$, $k = 1, \ldots, n$ (the sequences node and weight, respectively), in the quadrature sum

$$Q_n(f; w) = \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}) \tag{2.12}$$

can be obtained in the Mathematica Package OrthogonalPolynomials by the command aGaussianNodesWeights, giving number of points n (n), the recurrence coefficients α_k and β_k (sequences alpha and beta), as well as the WorkingPrecision and Precision by numerical parameters WP and PR, respectively (usually we put PR=WP-5 or PR=WP-10). For example, for $\alpha = 1/2$, $N = 100$, by the following commands we obtain the parameters of the n-point quadrature formula (2.12), for $n = 20$ and $n = 100$, with precision of PR =230 decimal digits,

```
<<orthogonalPolynomials`
  mom200=Table[If[OddQ[k],0,2/((1+k)(3+2k))], {k,0,199}];
  {alpha,beta}=aChebyshevAlgorithm[mom200,Algorithm->Symbolic];
  nw[n_]:=aGaussianNodesWeights[n,alpha,beta,WorkingPrecision->240, Precision->230];
  {node20,weight20}=nw[20]; {node100,weight100}=nw[100];
```

Example 2.1 We consider a simple weighted integral on $(-1, 1)$, given by

$$I = I(F; 1) = I(f; w) = \int_{-1}^{1} \frac{1-\sqrt{|x|}}{x(x+1)} \sin 4\pi x \, dx, \tag{2.13}$$

with the weight function $w(x) = B_{1/2}(x) = 1 - \sqrt{|x|}$. Here

$$F(x) = \frac{1-\sqrt{|x|}}{x(x+1)} \sin 4\pi x \quad \text{and} \quad f(x) = \frac{\sin 4\pi x}{x(x+1)},$$

and their graphics are presented in Fig. 5.

Its exact value can be expressed in terms of the sine integral function $x \mapsto \mathrm{Si}(x) = \int_0^x (\sin t/t)\, dt$ and the hypergeometric function ${}_1F_2$ as

$$I = 2\mathrm{Si}(4\pi) - \mathrm{Si}(8\pi) - 4\pi \frac{\mathrm{d}}{\mathrm{d}c} \left\{ {}_1F_2\left(\begin{matrix} 3/4 \\ 3/2 \quad c \end{matrix} \,\middle|\, -4\pi^2 \right) \right\} \Bigg|_{c=3/4},$$

with the numerical value $I = 2.39893583689780749749151598817332198435995\ldots$.

For testing a quadrature rule $Q_n(f; w)$ for computing the integral $I = I(f; w)$ we use the relative error

$$\mathrm{Err}[Q_n(f; w)] = \left| \frac{Q_n(f; w) - I(f; w)}{I(f; w)} \right|. \tag{2.14}$$

Because of the critical singularity at the origin $x = 0$ in the integrand $F(x)$ in (2.13), the standard Gauss-Legendre quadrature formula cannot be successfully applied in this example, because the convergence of the Gauss-Legendre quadrature sum $Q_n(F; 1)$ is very slow. In Table 1 we give the relative errors in the Gaussian quadrature sums $Q_n(F; 1)$ (Gauss-Legendre) for $n = 5(5)30, 40, 50$, and 100. Numbers in parentheses indicate decimal exponents. As we can see these quadrature sums give only two exact decimal digits of the integral, even using 100-point Gauss-Legendre rule.

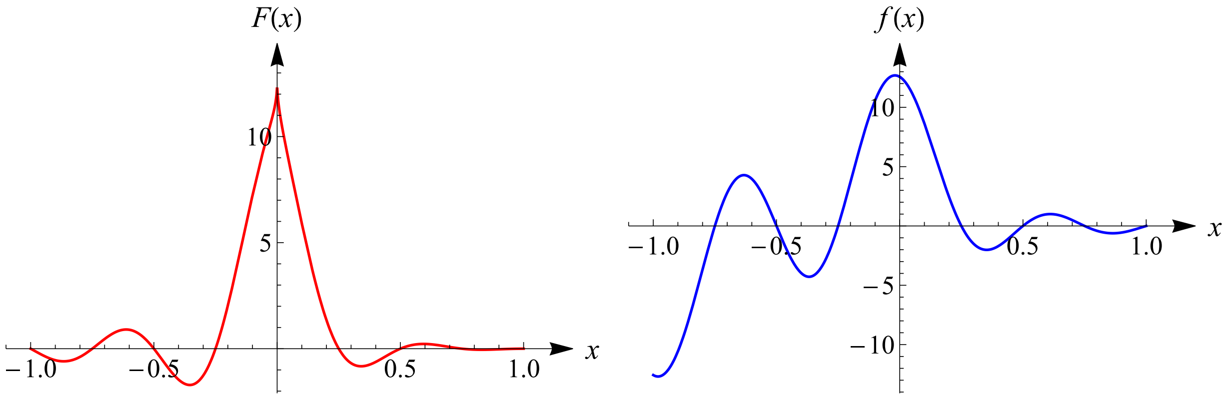

Figure 2. Graphics of the integrand $x \mapsto F(x) = B_{1/2}(x)f(x)$ (left) and the function $x \mapsto f(x)$ (right)

Table 1. Relative errors $\mathrm{Err}[Q_n(F;1)]$ in Gauss-Legendre sums and $\mathrm{Err}[Q_n(f;B_{1/2})]$ in Gaussian sums of the quadrature rule $Q_n(f;w)$

n	$\mathrm{Err}[Q_n(F;1)]$	$\mathrm{Err}[Q_n(f;B_{1/2})]$
5	2.06	3.13(−1)
10	1.45(−1)	1.92(−5)
15	2.08(−1)	8.21(−12)
20	4.12(−2)	1.43(−19)
25	9.57(−2)	2.24(−28)
30	2.18(−2)	5.14(−38)
40	1.40(−2)	2.71(−59)
50	1.00(−2)	1.08(−82)
100	3.54(−3)	1.28(−220)

Alternatively, we now apply the Gaussian quadrature formula constructed for the weight function $w(x) = B_{1/2}(x) = 1 - \sqrt{|x|}$.

In the same Table 1 we give the corresponding relative errors $\mathrm{err}[Q_n(f;B_{1/2})]$ computed by (2.14), where the quadrature sums $Q_n(f;w)$ are given by (2.12) for Gaussian quadrature rule (2.8). We can see that our quadrature formula of Gaussian type (2.8) with respect to the weight function $w(x) = B_{1/2}(x) = 1 - \sqrt{|x|}$ converges very fast. With only $n = 20$ nodes the obtained result has about 19 exact decimal digits (relative error is of order 10^{-19}), and for $n = 100$ nodes the number of exact decimal digits is about 220!

3. Orthogonal Polynomials and Gaussian Quadrature Rules with Respect to the Weight Function (1.7)

In this section we consider a more general case (1.7), i.e., when

$$w(x) = W_{\alpha,\beta}(x) = |x|^{2/\beta-1}(1-|x|^{2/\beta})^{\alpha-1}, \quad \alpha, \beta > 0, \tag{3.1}$$

on $[-1,1]$. Evidently, for $\alpha = \beta = 2$ it reduces to the weight $B_1(x) = 1 - |x|$. For $\beta = 2$ and different value of α, the graphics of $x \mapsto W_{\alpha,2}(x)$ are presented in Fig. 3.

The moments of the general two-parametric even weight function (1.7), i.e., (3.1), are

$$\mu_k = \int_{-1}^{1} x^k W_{\alpha,\beta}(x)\,\mathrm{d}x = \begin{cases} \dfrac{\beta\,\Gamma(\alpha)\Gamma\left(1+\frac{1}{2}\beta k\right)}{\Gamma\left(1+\alpha+\frac{1}{2}\beta k\right)}, & k\,(\geq 0) \text{ is even,} \\ 0, & k\,(\geq 1) \text{ is odd.} \end{cases} \tag{3.2}$$

As before, the corresponding (monic) orthogonal polynomials $\pi_k(\,\cdot\,) \equiv \pi_k(\,\cdot\,; W_{\alpha,\beta})$ satisfy the three-term recurrence relation of the form

$$\pi_{k+1}(x) = x\pi_k(x) - \beta_k\pi_{k-1}(x), \quad k = 1, 2, \ldots, \tag{3.3}$$

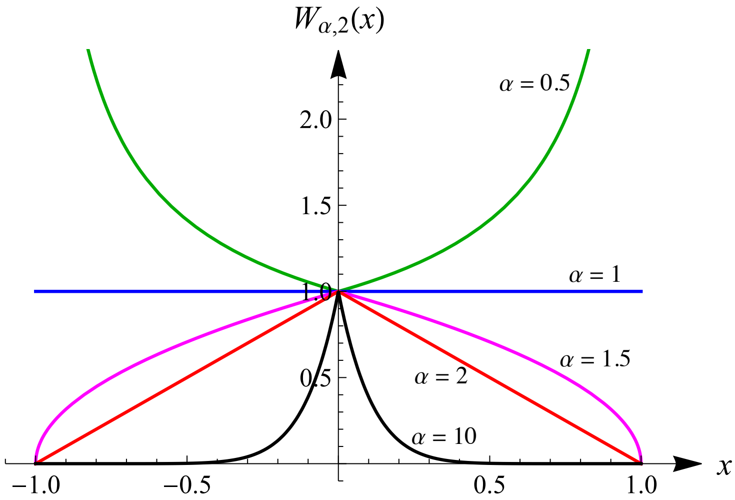

Figure 3. Graphics of the weight function $x \mapsto W_{\alpha,2}(x)$ for $\alpha = 1/2, 1, 3/2, 2,$ and 10

with β-coefficients

$$\beta_0 = \frac{\beta}{\alpha}, \quad \beta_1 = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)},$$

$$\beta_2 = \frac{\Gamma(2\beta+1)\Gamma(\alpha+\beta+1)^2 - \Gamma(\alpha+1)\Gamma(\beta+1)^2\Gamma(\alpha+2\beta+1)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+1)\Gamma(\alpha+2\beta+1)},$$

$$\beta_3 = \frac{\Gamma(\alpha+\beta+1)^2\left[\Gamma(\beta+1)\Gamma(3\beta+1)\Gamma(\alpha+2\beta+1)^2 - \Gamma(2\beta+1)^2\Gamma(\alpha+\beta+1)\Gamma(\alpha+3\beta+1)\right]}{\Gamma(\beta+1)\Gamma(\alpha+2\beta+1)\Gamma(\alpha+3\beta+1)\left[\Gamma(2\beta+1)\Gamma(\alpha+\beta+1)^2 - \Gamma(\alpha+1)\Gamma(\beta+1)^2\Gamma(\alpha+2\beta+1)\right]},$$

etc.

Remark 3.1 Because of positivity of β-coefficients for nonnegative weight functions, we conclude that the following inequalities

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} > \sqrt{\frac{\Gamma(\alpha+1)\Gamma(\alpha+2\beta+1)}{\Gamma(2\beta+1)}} \quad \text{and} \quad \frac{\Gamma(\alpha+2\beta+1)}{\Gamma(2\beta+1)} > \sqrt{\frac{\Gamma(\alpha+\beta+1)\Gamma(\alpha+3\beta+1)}{\Gamma(\beta+1)\Gamma(3\beta+1)}}$$

hold for each $\alpha, \beta > 0$.

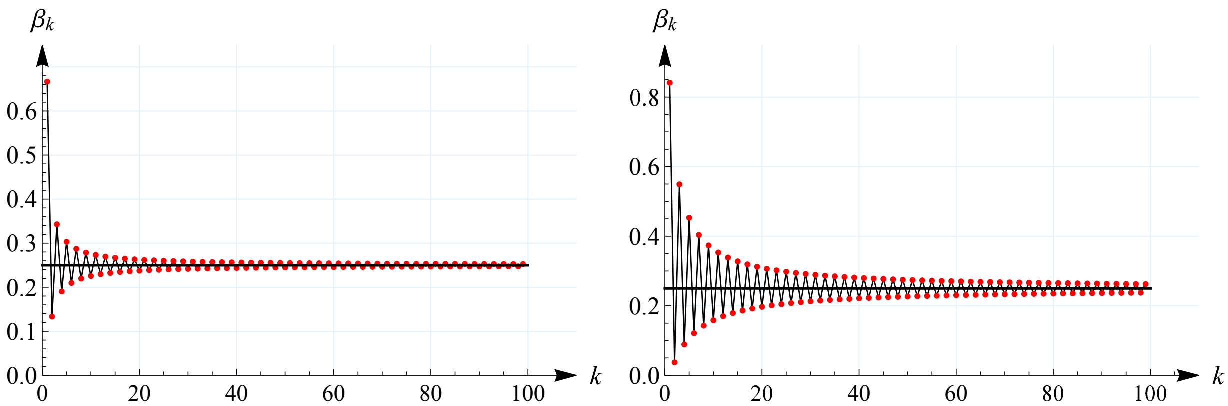

Figure 4. The recurrence coefficients β_k, $k = 1, 2, \ldots, 99$, for polynomials orthogonal on $(-1, 1)$ with respect to the weight functions $x \mapsto W_{1/2,1}(x) = |x|/\sqrt{1-x^2}$ (left) and $x \mapsto W_{1/2,1/3}(x) = |x|^5/\sqrt{1-x^6}$ (right)

These parameters β_k were obtained by the routine `aChebyshevAlgorithm`, with the option `Algorithm->Symbolic`, using our Mathematica Package `OrthogonalPolynomials` (see [11], [26]).

In order to get the first $N = 5$ coefficients in symbolic form it needs 21 ms, but for $N = 10$ this time is about 3 seconds. Any further increase in the number of coefficients requires an exponential increase in time, e.g. for $N = 11$, 12, and 13, these times are 8″, 24″, and 1′24″, respectively. However, if we decide to use numerical option in `aChebyshevAlgorithm`, for a fixed values of α and β, with a given `WorkingPrecision` (`WP`), we can construct the recurrence coefficients very fast. For example, if we take $\alpha = 1/2$ and $\beta = 1$, the first $N = 100$ recursive coefficients can be obtained with the maximal relative error of $9.37(-24)$ in only 63 ms, taking `WP=80`. These coefficients β_k, $k = 1, 2, \ldots, 99$, are presented in Fig. 4 (left). We can see that the sequence $\{\beta_{2k-1}\}$ is decreasing, and $\{\beta_{2k}\}$ is increasing, but so that $\lim_{k\to\infty} \beta_k = 1/4$.

Remark 3.2 The previous weight function is $W_{1/2,1}(x) = |x|/\sqrt{1-x^2}$ is a special case considered by Laščenov [22], whose recurrence coefficients given by (2.5) and (2.6). Thus, in this case we have

$$\beta_k = \begin{cases} \dfrac{k(k+1)}{4k^2-1}, & k\ (\geq 1) \text{ is odd}, \\[2mm] \dfrac{k(k-1)}{4k^2-1}, & k\ (\geq 2) \text{ is even}. \end{cases}$$

and $\beta_0 = 2$. This sequence is given by

$$\{\beta_k\}_{k=1}^{\infty} = \left\{\frac{2}{3}, \frac{2}{15}, \frac{12}{35}, \frac{4}{21}, \frac{10}{33}, \frac{30}{143}, \frac{56}{195}, \frac{56}{255}, \frac{90}{323}, \frac{30}{133}, \frac{44}{161}, \frac{132}{575}, \frac{182}{675}, \frac{182}{783}, \ldots\right\}.$$

In the following strong nonclassical case of (3.1), with parameters $\alpha = 1/2$ and $\beta = 1/3$, we have the weight function

$$W_{1/2,1/3}(x) = \frac{|x|^5}{\sqrt{1-x^6}}. \tag{3.4}$$

As before, numerical construction of recurrence coefficients by the Mathematica Package `OrthogonalPolynomials` (see [11], [26]) is very fast. The first $N = 100$ recurrence coefficients can be obtained with the maximal relative error of $5.22(-18)$ in only 56 ms, taking `WP=80`. These coefficients β_k, $k = 1, 2, \ldots, 99$, are presented in Fig. 4 (right).

4. Orthogonality on $(0, 1)$ and Quadrature Formulas for Fractional Integrals

A class of quasi-polynomials orthogonal with respect to the fractional integration operator has been developed in [29], as well as the related quadrature formulas of Gaussian type. In fact, the authors in [29] considered the problem of numerical evaluation of the left fractional integral for $a = 0$ (see (1.5))

$$_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1}\, \mathrm{d}\tau, \tag{4.1}$$

when $t = 1$, introducing a family of (monic) β-polynomials

$$P_{n,\beta}^{(\alpha)}(t) = \sum_{\nu=0}^{n} c_{n,\nu}^{(\alpha,\beta)} t^{\nu\beta} \quad (c_{n,n}^{(\alpha,\beta)} = 1),$$

orthogonal in the sense that

$$_0I_1^{\alpha}\big(P_{n,\beta}^{(\alpha)} P_{m,\beta}^{(\alpha)}\big) = 0, \quad n \neq m. \tag{4.2}$$

Their result [29] can be expressed in the following form:

Theorem 4.1 *There exists a family of β-polynomials $P_{n,\beta}^{(\alpha)}(t) \equiv P_n^{(\alpha,\beta)}(x)$, $x = t^{\beta}$, satisfying (4.2). These quasi-polynomials can be obtained recursively by means of*

$$P_{k+1}^{(\alpha,\beta)}(x) = \big(x - A_k^{(\alpha,\beta)}\big) P_k^{(\alpha,\beta)}(x) - B_k^{(\alpha,\beta)} P_{k-1}^{(\alpha,\beta)}(x), \quad P_0^{(\alpha,\beta)}(x) = 1, \tag{4.3}$$

with the recurrence coefficients given by Darboux's formulas

$$A_k^{(\alpha,\beta)} = \frac{\langle xP_k^{(\alpha,\beta)}, P_k^{(\alpha,\beta)}\rangle_w}{\langle P_k^{(\alpha,\beta)}, P_k^{(\alpha,\beta)}\rangle_w}, \quad B_k^{(\alpha,\beta)} = \frac{\langle P_k^{(\alpha,\beta)}, P_k^{(\alpha,\beta)}\rangle_w}{\langle P_{k-1}^{(\alpha,\beta)}, P_{k-1}^{(\alpha,\beta)}\rangle_w},$$

where the inner product $\langle \cdot, \cdot \rangle_w$ is defined by (1.1) on $(a, b) = (0, 1)$, with the weight function

$$w(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)} \frac{(1-x^{1/\beta})^{\alpha-1}}{\beta x^{1-1/\beta}}, \quad \alpha, \beta > 0. \tag{4.4}$$

Remark 4.1 In the mentioned paper [29], the authors listed a few first monic orthogonal polynomials (for $\beta = 1$):

$$P_{0,1}^{(\alpha)}(t) = 1, \quad P_{1,1}^{(\alpha)}(t) = t - \frac{1}{\alpha+1}, \quad P_{2,1}^{(\alpha)}(t) = t^2 - \frac{4t}{\alpha+3} + \frac{2}{(\alpha+2)(\alpha+3)},$$

$$P_{3,1}^{(\alpha)}(t) = t^3 - \frac{9t^2}{\alpha+5} + \frac{18t}{(\alpha+4)(\alpha+5)} - \frac{6}{(\alpha+3)(\alpha+4)(\alpha+5)}, \quad \ldots,$$

as well as a few quasi-polynomials (with different the so-called commensurate order β and $\alpha = 1$) $P_{n,\beta}^{(1)}(t)$, i.e., $P_n^{(1,\beta)}(x)$ ($x = t^\beta$):

$$P_0^{(1,\beta)}(x) = 1, \quad P_1^{(1,\beta)}(x) = x - \frac{1}{\beta+1}, \quad P_2^{(1,\beta)}(x) = x^2 - \frac{2(\beta+1)x}{3\beta+1} + \frac{\beta+1}{(2\beta+1)(3\beta+1)},$$

$$P_3^{(1,\beta)}(x) = x^3 - \frac{3(2\beta+1)x^2}{5\beta+1} + \frac{3(\beta+1)(2\beta+1)x}{(4\beta+1)(5\beta+1)} - \frac{(\beta+1)(2\beta+1)}{(3\beta+1)(4\beta+1)(5\beta+1)},$$

$$P_4^{(1,\beta)}(t) = t^4 - \frac{4(3\beta+1)t^3}{7\beta+1} + \frac{6(2\beta+1)(3\beta+1)t^2}{(6\beta+1)(7\beta+1)} - \frac{4(\beta+1)(2\beta+1)(3\beta+1)t}{(5\beta+1)(6\beta+1)(7\beta+1)} + \frac{(\beta+1)(2\beta+1)(3\beta+1)}{(4\beta+1)(5\beta+1)(6\beta+1)(7\beta+1)},$$

etc.

The polynomials $P_k^{(\alpha,\beta)}(x)$ from Theorem 4.1 can be connected by polynomials $\pi_k(\,\cdot\,) \equiv \pi_k(\,\cdot\,; W_{\alpha,\beta})$, which satisfy the three-term recurrence relation (3.3). The weight function $x \mapsto W_{\alpha,\beta}(x)$ is defined on $(-1,1)$ by (3.1).

Theorem 4.2 *If the monic orthogonal polynomials $\pi_k(\,\cdot\,) \equiv \pi_k(\,\cdot\,; W_{\alpha,\beta})$, with parameters $\alpha, \beta > 0$, satisfy the three-term recurrence relation (3.3), then*

1° *the polynomials $P_k^{(\alpha,\beta)}(x) = \pi_{2k}\left(\sqrt{x}\right)$ are orthogonal on $(0,1)$ with respect to the weight function (4.4), i.e., $x \mapsto x^{1/\beta-1}(1-x^{1/\beta})^{\alpha-1}$, and satisfy the three-term recurrence relation (4.3), with the coefficients given by*

$$A_0^{(\alpha,\beta)} = \beta_1, \quad A_k^{(\alpha,\beta)} = \beta_{2k} + \beta_{2k+1}, \quad B_k^{(\alpha,\beta)} = \beta_{2k-1}\beta_{2k}.$$

2° *the monic polynomials $\tilde{P}_k^{(\alpha,\beta)}(x) = \pi_{2k+1}\left(\sqrt{x}\right)/\sqrt{x}$ are orthogonal on $(0,1)$ with respect to the weight function $x \mapsto x^{1/\beta}(1-x^{1/\beta})^{\alpha-1}$, and satisfy the three-term recurrence relation of the form (4.3), with the corresponding coefficients given by*

$$\tilde{A}_0^{(\alpha,\beta)} = \beta_1 + \beta_2, \quad \tilde{A}_k^{(\alpha,\beta)} = \beta_{2k+1} + \beta_{2k+2}, \quad \tilde{B}_k^{(\alpha,\beta)} = \beta_{2k}\beta_{2k+1}.$$

Proof. See Theorems 2.2.11 and 2.2.12 in [23, pp. 102-103]. □

In special cases given in Remark 4.1, the polynomials $P_{n,\beta}^{(\alpha)}(t)$ for $\beta = 1$, as well as ones for $\alpha = 1$, can be expressed in the explicit forms.

Corollary 4.1 *We have*

$$P_{n,1}^{(\alpha)}(t) = P_n^{(\alpha,1)}(t) = \sum_{\nu=0}^{n} (-1)^\nu \frac{\nu!}{(2n-\nu+\alpha)_\nu} \binom{n}{\nu}^2 t^{n-\nu}, \quad n = 0, 1, \ldots, \tag{4.5}$$

where $(a)_\nu$ denotes the Pochhammer symbol defined by

$$(a)_\nu = \begin{cases} a(a+1)\cdots(a+\nu-1), & \nu \in \mathbb{N}; \\ 1, & \nu = 0. \end{cases}$$

The corresponding recurrence coefficients are

$$A_k^{(\alpha,1)} = \frac{2k^2 + 2\alpha k + \alpha - 1}{(2k+\alpha)^2 - 1}, \quad k = 0, 1, 2, \ldots,$$

and

$$B_0^{(\alpha,1)} = \frac{1}{\Gamma(\alpha+1)}, \quad B_k^{(\alpha,1)} = \frac{k^2(k-1+\alpha)^2}{(2k-2+\alpha)(2k-1+\alpha)^2(2k+\alpha)}, \quad k = 1, 2, \ldots.$$

Proof. In this case the weight function is given by $x \mapsto (1-x)^{\alpha-1}/\Gamma(\alpha)$ on $(0,1)$. The coefficients $A_k^{(\alpha,1)}$ and $B_k^{(\alpha,1)}$ can be obtained from (2.3), taking $n := k$, $\beta := 0$, and $\alpha := \alpha - 1$. In addition,

$$B_0^{(\alpha,1)} = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1}\,\mathrm{d}x = \frac{1}{\alpha\Gamma(\alpha)}.$$

The expression (4.5) for $P_n^{(\alpha,1)}(t)$ $(= P_{n,1}^{(\alpha)}(t))$ can be proved by induction, using the corresponding three-term recurrence relation. □

In the sequel we need the following auxiliary result:

Lemma 4.1 *For $n \in \mathbb{N}$, $0 \le k \le n$, and each $a \in \mathbb{C}$ we have*

$$\sum_{\nu=0}^{k} \frac{(-1)^\nu \nu!}{(2n-\nu+a)_\nu} \binom{k}{\nu}\binom{n}{\nu} = \frac{(n-k+a)_k}{(2n-k+a)_k}, \quad 0 \le k \le n.$$

Proof. Using the classical Gauss summation formula from 1812 [4, p. 66] (see also [19, 27]),

$${}_2F_1\left[\begin{matrix}\alpha_1,\ \alpha_2\\ \beta_1\end{matrix}\,\middle|\, 1\right] = \sum_{\nu=0}^{\infty} \frac{(\alpha_1)_\nu(\alpha_2)_\nu}{(\beta_1)_\nu}\cdot\frac{1}{\nu!} = \frac{\Gamma(\beta_1)\Gamma(\beta_1-\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)},$$

with $\alpha_1 = -k$, $\alpha_2 = -n$, $\beta_1 = -2n+1-a$ and $z = 1$, where $k, n \in \mathbb{N}_0$ and $k \le n$, the previous sum reduces to a finite sum

$$\sum_{\nu=0}^{k} \frac{(-k)_\nu(-n)_\nu}{\nu!(-2n+1-a)_\nu} = \frac{\Gamma(-2n+1-a)\Gamma(-n+k+1-a)}{\Gamma(-2n+k+1-a)\Gamma(-n+1-a)} = \frac{(-n+1-a)_k}{(-2n+1-a)_k}.$$

Since

$$\binom{k}{\nu} = \frac{(-1)^\nu}{\nu!}(-k)_\nu, \quad \binom{n}{\nu} = \frac{(-1)^\nu}{\nu!}(-n)_\nu, \quad (-2n+1-a)_\nu = (-1)^\nu(2n-\nu+a)_\nu,$$

we conclude that the identity

$$\sum_{\nu=0}^{k} \frac{(-1)^\nu \nu!}{(2n-\nu+a)_\nu} \binom{k}{\nu}\binom{n}{\nu} = \frac{(n-k+a)_k}{(2n-k+a)_k}$$

holds for each $0 \le k \le n$ and $a \in \mathbb{C}$. □

Corollary 4.2 *The recurrence coefficients for the polynomials $P_n^{(1,\beta)}(x)$ are*

$$A_k^{(1,\beta)} = \frac{1+(2k-1)\beta+2k^2\beta^2}{[1+(2k-1)\beta][1+(2k+1)\beta]}, \quad k = 0,1,2,\ldots,$$

and

$$B_0^{(1,\beta)} = \beta, \quad B_k^{(1,\beta)} = \frac{k^2\beta^2[1+(k-1)\beta]^2}{[1+(2k-2)\beta][1+(2k-1)\beta]^2[1+2k\beta]}, \quad k = 1,2,\ldots,$$

and its explicit expression can be given in the form

$$P_n^{(1,\beta)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \prod_{\nu=1}^{k} \frac{(n-\nu)\beta+1}{(2n-\nu)\beta+1}\, x^{n-k}, \tag{4.6}$$

where the empty product (for $k = 0$) is equal to 1.

Proof. The polynomials $P_n^{(1,\beta)}(x)$ are orthogonal with respect to the weight function $x \mapsto x^{1/\beta-1}$ on $(0,1)$. If we make changes $t := 1-x$ and $\alpha := 1/\beta$ in Corollary 4.1, we get

$$\begin{aligned}
P_n^{(1,\beta)}(x) &= (-1)^n P_{n,1}^{(1/\beta)}(1-x)\\
&= \sum_{\nu=0}^{n} (-1)^{n-\nu} \frac{\nu!}{(2n-\nu+1/\beta)_\nu}\binom{n}{\nu}^2 (1-x)^{n-\nu}\\
&= \sum_{\nu=0}^{n} (-1)^{n-\nu} \frac{\nu!}{(2n-\nu+1/\beta)_\nu}\binom{n}{\nu}^2 \sum_{k=0}^{n-\nu} (-1)^k \binom{n-\nu}{k} x^k.
\end{aligned}$$

According to the property

$$\sum_{\nu=0}^{n}\sum_{k=0}^{n-\nu} A_{\nu,k} = \sum_{k=0}^{n}\sum_{\nu=0}^{k} A_{\nu,n-k},$$

we have

$$\begin{aligned} P_n^{(1,\beta)}(x) &= \sum_{k=0}^{n}(-1)^k x^{n-k}\sum_{\nu=0}^{k}(-1)^\nu \frac{\nu!}{(2n-\nu+1/\beta)_\nu}\binom{n}{\nu}^2\binom{n-\nu}{n-k} \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\left\{\sum_{\nu=0}^{k}(-1)^\nu\frac{\nu!}{(2n-\nu+1/\beta)_\nu}\binom{k}{\nu}\binom{n}{\nu}\right\}x^{n-k}\end{aligned}$$

The expression in curly braces $S_k^{(n)}(\beta)$ becomes

$$S_k^{(n)}(\beta) = \sum_{\nu=0}^{k}(-1)^\nu \nu!\binom{k}{\nu}\binom{n}{\nu}\frac{1}{(2n-\nu+1/\beta)_\nu}.$$

Using Lemma 4.1 (with $a = 1/\beta$) we obtain that

$$S_k^{(n)}(\beta) = \frac{(n-k+1/\beta)_k}{(2n-k+1/\beta)_k} = \prod_{\nu=1}^{k}\frac{(n-\nu)\beta+1}{(2n-\nu)\beta+1},$$

and for $k = 0$, $S_0^{(n)}(\beta) = 1$. This proves (4.6).

Using the recurrence relation (4.3) and the property

$$P_k^{(1,\beta)}(x) = (-1)^k P_k^{(1/\beta,1)}(1-x),$$

as well as Corollary 4.1 we obtain $A_k^{(1,\beta)} = 1 - A_k^{(1/\beta,1)}$ and $B_k^{(1,\beta)} = B_k^{(1/\beta,1)}$. $\square$

4.1. *The case* (4.1) *for* $\alpha = \beta = 1/2$

For these parameters the weight function becomes

$$w\left(x;\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{2}{\sqrt{\pi}}\cdot\frac{x}{\sqrt{1-x^2}}, \tag{4.7}$$

where the numerical factor $2/\sqrt{\pi}$ is not important, and it will be omitted in the sequel. This is one-side variant of the even weight function $W_{1/2,1}(x)$ considered in Remark 3.2.

The moments of the weight function (4.7) are

$$\mu_k = \int_0^1 x^k w\left(x;\tfrac{1}{2},\tfrac{1}{2}\right)\mathrm{d}x = \frac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k+3}{2}\right)} = \begin{cases}\dfrac{2^{k+1}}{(k+1)\binom{k}{k/2}\sqrt{\pi}}, & k\,(\geq 0)\ \text{is even},\\[2ex] \dfrac{\sqrt{\pi}\binom{k}{(k-1)/2}}{2^k} & k\,(\geq 1)\ \text{is odd},\end{cases}$$

i.e.,

$$\left\{\frac{2}{\sqrt{\pi}},\frac{\sqrt{\pi}}{2},\frac{4}{3\sqrt{\pi}},\frac{3\sqrt{\pi}}{8},\frac{16}{15\sqrt{\pi}},\frac{5\sqrt{\pi}}{16},\frac{32}{35\sqrt{\pi}},\frac{35\sqrt{\pi}}{128},\frac{256}{315\sqrt{\pi}},\frac{63\sqrt{\pi}}{256},\frac{512}{693\sqrt{\pi}},\frac{231\sqrt{\pi}}{1024},\frac{2048}{3003\sqrt{\pi}},\ldots\right\}.$$

For the corresponding recurrence coefficients, using the Mathematica Package OrthogonalPolynomials (see [11], [26]), we obtain

$$\alpha_0 = \frac{\pi}{4},\quad \alpha_1 = \frac{\pi\left(3\pi^2-28\right)}{4\left(32-3\pi^2\right)},\quad \alpha_2 = \frac{\pi\left(34816-7524\pi^2+405\pi^4\right)}{4\left(32-3\pi^2\right)\left(2048-207\pi^2\right)},$$

$$\alpha_3 = \frac{9\pi\left(822083584-270065664\pi^2+29416500\pi^4-1063125\pi^6\right)}{4\left(8388608-1549440\pi^2+70875\pi^4\right)\left(207\pi^2-2048\right)},$$

$$\alpha_4 = \frac{3\pi A}{4\left(8388608-1549440\pi^2+70875\pi^4\right)\left(137438953472-27709286400\pi^2+1396591875\pi^4\right)},\quad \ldots,$$

where

$$A = 133559876449206272 - 583026471862272000\pi^2 + 9499436559360000\pi^4 - 685081896187500\pi^6 + 18460501171875\pi^8,$$

and

$$\beta_0 = \frac{2}{\sqrt{\pi}}, \quad \beta_1 = \frac{1}{48}\left(32 - 3\pi^2\right), \quad \beta_2 = -\frac{207\pi^2 - 2048}{15\left(3\pi^2 - 32\right)^2},$$

$$\beta_3 = -\frac{3\left(3\pi^2 - 32\right)\left(8388608 - 1549440\pi^2 + 70875\pi^4\right)}{560\left(207\pi^2 - 2048\right)^2},$$

$$\beta_4 = -\frac{\left(207\pi^2 - 2048\right)\left(137438953472 - 27709286400\pi^2 + 1396591875\pi^4\right)}{21\left(8388608 - 1549440\pi^2 + 70875\pi^4\right)^2},$$

etc. For constructing the first 20 (50) recurrence coefficients in symbolic form, using our Mathematica package OrthogonalPolynomials, we need 0.6 (40.7) seconds.

However, in numerical mode this construction is very fast. For constructing the first 50 recurrence coefficients with about 22 exact decimal digits, i.e., when the maximal relative error in these coefficients

$$\max_{0\le k\le 49}\left\{\left|\frac{\alpha_k - \widehat{\alpha}_k}{\widehat{\alpha}_k}\right|, \left|\frac{\beta_k - \widehat{\beta}_k}{\widehat{\beta}_k}\right|\right\} \approx 5.82 \times 10^{-23},$$

we need the WorkingPrecision (WP=80) and only 16 ms. Here the exact values of the desired recurrence coefficients are denoted by $\widehat{\alpha}_k$ and $\widehat{\beta}_k$ and their values can be obtained using the same procedure, but with the higher working precision WP1 (e.g., with WP1=2 WP). If we use WP=100, then we obtain recurrence coefficients with maximal relative error 4.90×10^{-43}.

5. Numerical Computation of Fractional Riemann-Liouville Integrals

In this section we return to the fractional integral (4.1)

$$ {}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1}\,\mathrm{d}\tau, \quad \alpha > 0, \tag{5.1}$$

Following [29] we reduce (5.1) to an integral on $(0, 1)$, taking transformation $\tau = tx^{1/\beta}$, $\beta > 0$, so that we get the weighted integral

$$ {}_0I_t^{\alpha} f(t) = \frac{t^{\alpha}}{\beta\Gamma(\alpha)} \int_0^1 f\big(tx^{1/\beta}\big)x^{1/\beta-1}(1 - x^{1/\beta})^{\alpha-1}\,\mathrm{d}x,$$

i.e.,

$$ {}_0I_t^{\alpha} f(t) = t^{\alpha} \int_0^1 f\big(tx^{1/\beta}\big) w(x;\alpha,\beta)\,\mathrm{d}x, \quad \alpha,\beta > 0, \tag{5.2}$$

where the weight function $x \mapsto w(x;\alpha,\beta)$ is given by (4.4).

The fractional integral (5.1), i.e., (5.2), can be approximated by the weighted Gaussian quadrature sum

$$ {}_0I_t^{\alpha} f(t) \approx t^{\alpha} \sum_{k=1}^{n} A_{n,k}(w) f\big(t\xi_{n,k}(w)^{1/\beta}\big), \tag{5.3}$$

where the nodes and the weights, $\xi_{n,k}(w)$ and $A_{n,k}(w)$, $k = 1, \ldots, n$, depend on the two-parametric weight function $x \mapsto w(x;\alpha,\beta)$, and they can be constructed using our Mathematica package OrthogonalPolynomials as it is explained in Section 2, immediately before Example 2.1.

This procedure, in general, can be successfully applied to the both fractional Riemann-Liouville integrals given by (1.5), i.e.,

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau \quad t > a, \tag{5.4}$$

and

$$ {}_tI_b^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} f(\tau)\,\mathrm{d}\tau \quad t < b. \tag{5.5}$$

For the first of them, the so-called left fractional Riemann-Liouville integral (5.4), after using the change of variables $\tau = a + (t-a)x$, and after that $x := x^{1/\beta}$, where $\beta > 0$, the integral (5.4) reduces to

$$\begin{aligned} {}_aI_t^\alpha f(t) &= \frac{(t-a)^\alpha}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1} f(a+(t-a)x\,\mathrm{d}x \\ &= \frac{(t-a)^\alpha}{\beta\Gamma(\alpha)} \int_0^1 x^{1/\beta-1}(1-x^{1/\beta})^{\alpha-1} f\big(a+(t-a)x^{1/\beta}\big)\,\mathrm{d}x \\ &= \int_0^1 F_a(x;t,\alpha,\beta) w(x;\alpha,\beta)\,\mathrm{d}x, \quad \alpha,\beta>0, \end{aligned}$$

where the weight function $x \mapsto w(x;\alpha,\beta)$ is the same as in (5.2) defined by (4.4), and

$$F_a(x;t,\alpha,\beta) = (t-a)^\alpha f\big(a+(t-a)x^{1/\beta}\big). \tag{5.6}$$

The right fractional Riemann-Liouville integral (5.5), in a similar way, can be reduced to the following form

$${}_tI_b^\alpha f(t) = \int_0^1 F_b(x;t,\alpha,\beta) w(x;\alpha,\beta)\,\mathrm{d}x, \quad \alpha,\beta>0, \tag{5.7}$$

where

$$F_b(x;t,\alpha,\beta) = (b-t)^\alpha f\big(b-(b-t)x^{1/\beta}\big). \tag{5.8}$$

Then the following result is obviously valid.

Theorem 5.1 *Let $w(x) \equiv w(x;\alpha,\beta) = (\beta\Gamma(\alpha))^{-1} x^{1/\beta-1}(1-x^{1/\beta})^{\alpha-1}$, $\alpha,\beta>0$, with the moments*

$$\mu_k = \int_0^1 x^k w(x;\alpha,\beta)\,\mathrm{d}x = \frac{\Gamma(\beta k+1)}{\Gamma(\alpha+\beta k+1)}, \quad k=0,1,\ldots\,. \tag{5.9}$$

For each $n \in \mathbb{N}$ there exists a unique quadrature formula of maximal degree of precision $2n-1$ (the Gauss-Christoffel rule),

$$I[\varphi;w] = \int_0^1 \varphi(x) w(x;\alpha,\beta)\,\mathrm{d}x = Q_n(\varphi;w) + R_n(\varphi;w), \tag{5.10}$$

where

$$Q_n(\varphi;w) = \sum_{k=1}^n A_{n,k}(\alpha,\beta)\varphi\big(\xi_{n,k}(\alpha,\beta)\big), \tag{5.11}$$

with the nodes $\xi_k = \xi_{n,k}(\alpha,\beta)$, $k=1,\ldots,n$, which are eigenvalues of the Jacobi matrix (1.3) associated with the weight function $x \mapsto w(x;\alpha,\beta)$ and $A_k = A_{n,k}(\alpha,\beta)$, $k=1,\ldots,n$, are the corresponding Christoffel numbers. The remainder term $R_n(x^k;w) = \mu_k - Q_n(x^k;w) = 0$ for each $k=0,1,\ldots,2n-1$.

Then for the fractional Riemann-Liouville integrals (5.4) and (5.5) we have

$${}_aI_t^\alpha f(t) = Q_n\left(F_a(\,\cdot\,;t,\alpha,\beta);w\right) + R_n(F_a;w) \tag{5.12}$$

and

$${}_tI_b^\alpha f(t) = Q_n\left(F_b(\,\cdot\,;t,\alpha,\beta);w\right) + R_n(F_b;w) \tag{5.13}$$

for each $n \in \mathbb{N}$, where the functions F_a and F_b are given by (5.6) and (5.8), while $R_n(F_a;w)$ and $R_n(F_b;w)$ are the corresponding remainder terms.

For each $f \in C[a,b]$ the sequences of quadrature sums $\{Q_n\left(F_a(\,\cdot\,;t,\alpha,\beta);w\right)\}_{n=1}^\infty$ and $\{Q_n\left(F_b(\,\cdot\,;t,\alpha,\beta);w\right)\}_{n=1}^\infty$ converge to ${}_aI_t^\alpha f(t)$ and ${}_tI_b^\alpha f(t)$, respectively, and their rate of convergence is determined by the properties of the function f. Error estimates of Gaussian rules for some important classes of functions and the rate of convergence of corresponding quadrature sums can be found in [23, §5.1.5].

Selecting the parameter β we can remove a critical singularity in the origin (if any) and accelerate the convergence of the quadrature sums in the previous approximation. In Subsection 5.1 we give a few numerical examples in order to illustrate this effect.

Some improvements in the approximation of fractional Riemann-Liouville integrals can be achieved by applying the Radau quadrature formula instead of the Gauss-Christoffel formula (5.11).

Theorem 5.2 *As in Theorem 5.1, let $w(x) \equiv w(x;\alpha,\beta)$ be a weight function with the moments μ_k given by (5.9). Let*

$$w_0(x) = xw(x;\alpha,\beta) \quad \text{and} \quad w_1(x) = (1-x)w(x;\alpha,\beta)$$

be weight functions, with the moments $\mu_k^{(0)} = \mu_{k+1}$ and $\mu_k^{(1)} = \mu_k - \mu_{k+1}$, respectively. Then for each $n \in \mathbb{N}$ there exist the Gauss-Christoffel rules

$$\int_0^1 \varphi(x)w_\nu(x)\,\mathrm{d}x = Q_n(\varphi;w_\nu) + R_n(\varphi;w_\nu) \quad (\nu = 0,1),$$

with quadrature sums and Gaussian parameters (nodes and weights)

$$Q_n(\varphi;w_\nu) = \sum_{k=1}^{n} A_k^{(\nu)}\varphi(\xi_k^{(\nu)}), \quad \xi_k^{(\nu)} = \xi_{n,k}^{(\nu)}(\alpha,\beta), \quad A_k^{(\nu)} = A_{n,k}^{(\nu)}(\alpha,\beta) \quad (\nu = 0,1),$$

as well as the Radau quadrature rules for the weighted integral (5.10) of the algebraic degree of exactness $2n$,

$$I[\varphi;w] = \int_0^1 \varphi(x)w(x)\,\mathrm{d}x = Q_n^{(\nu)}(\varphi) + R_n^{(\nu)}(\varphi;w) \quad (\nu = 0,1),$$

where

$$Q_n^{(0)}(\varphi) = B_0\varphi(0) + \sum_{k=1}^{n} B_k\varphi(\xi_k^{(0)}) \quad \text{and} \quad Q_n^{(1)}(\varphi) = \sum_{k=1}^{n} C_k\varphi(\xi_k^{(1)}) + C_0\varphi(1),$$

with quadrature parameters

$$B_k = \frac{A_k^{(0)}}{\xi_k^{(0)}} \quad (k = 1,2,\ldots,n), \quad B_0 = \mu_0 - \sum_{k=1}^{n} B_k$$

and

$$C_k = \frac{A_k^{(1)}}{1-\xi_k^{(1)}} \quad (k = 1,2,\ldots,n), \quad C_0 = \mu_0 - \sum_{k=1}^{n} C_k.$$

$R_n^{(\nu)}(\varphi;w)$, $\nu = 0,1$, are the corresponding remainder terms.

Then for the fractional Riemann-Liouville integrals (5.4) and (5.5) we have

$$ {}_aI_t^\alpha f(t) = Q_n^{(1)}\left(F_a(\,\cdot\,;t,\alpha,\beta);w\right) + R_n^{(1)}(F_a;w) \tag{5.14}$$

and

$$ {}_tI_b^\alpha f(t) = Q_n^{(0)}\left(F_b(\,\cdot\,;t,\alpha,\beta);w\right) + R_n^{(0)}(F_b;w) \tag{5.15}$$

for each $n \in \mathbb{N}$, where the functions F_a and F_b are given by (5.8) and (5.6), while $R_n^{(1)}(F_a;w)$ and $R_n^{(0)}(F_b;w)$ are the corresponding remainder terms.

Theorems 5.1 and 5.2 give two efficient procedures for numerical computation of the left and right fractional Riemann-Liouville integrals.

5.1. Numerical examples

In order to illustrate the efficiency of our method for calculating fractional Riemann-Liouville integrals we give a few examples. With $QS_n^{(\alpha,\beta)}[f;t]$ we denote one of the quadrature sums obtained by quadrature formulas (5.12), (5.13), (5.14), and (5.15), with respect to the weight function $x \mapsto w(x;\alpha,\beta)$ on $[0,1]$ defined by (4.4).

In all examples we calculate the relative errors as in (2.14),

$$\mathrm{Err}_n^{(\alpha,\beta)} f(t) = \left|\frac{QS_n^{(\alpha,\beta)}[f;t] - I[f;t]}{I[f;t]}\right| \qquad (a \le t \le b), \tag{5.16}$$

where $I[f;t]$ is the exact value of one of the fractional integrals ${}_aI_t^\alpha f(t)$ and ${}_tI_b^\alpha f(t)$, given by (5.4) and (5.4), respectively. We calculate the relative errors (5.16) at the selected points

$$t = t_\nu = a + (b-a)\frac{\nu}{100}, \qquad \nu = 0,1,2,\ldots,100,$$

taking the n-point quadrature rule. We usually in our examples show every fifth point in the graphics or give it as a continuous curve by connecting all the points.

Example 5.1 We consider the left fractional integral (5.1), with a function as in [29], i.e., when

$$f(t) = \mathrm{e}^t \mathrm{erfc}\big(\sqrt{t}\big) = E_{1/2,1}\big(-\sqrt{t}\big), \tag{5.17}$$

whose exact solution can be obtained by using the Laplace transform. Here, $z \mapsto \mathrm{erfc}(z) = 1 - \mathrm{erf}(z)$ is the so-called complementary error function of the integral of the Gaussian distribution

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\,\mathrm{d}t,$$

and $E_{\alpha,\beta}(z)$ is two-parametric Mittag-Leffler function [17, Chapter 4], [18, p. 42] (see also [3]), defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Applying the Laplace transform to the convolution integral (5.1) we get

$$\mathcal{L}[{}_0I_t^{\alpha} f(t)] = \frac{1}{\Gamma(\alpha)} \mathcal{L}\big[f(t) \star t^{\alpha-1}\big] = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\sqrt{s}+s} \cdot \frac{\Gamma(\alpha)}{s^{\alpha}} = \frac{1}{s^{\alpha+1/2}(1+\sqrt{s})},$$

from which the exact fractional integral is given by

$${}_0I_t^{\alpha} f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+1/2}(1+\sqrt{s})}\right] = \frac{t^{\alpha} - \mathrm{e}^t \Gamma(\alpha+1,t)}{\Gamma(\alpha+1)} + \frac{\mathrm{e}^t \Gamma\left(\alpha+\frac{1}{2},t\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)}, \tag{5.18}$$

where $\Gamma(a,z) = \int_z^{\infty} t^{a-1} \mathrm{e}^{-t}\,\mathrm{d}t$ denotes the incomplete gamma function. The graphics of the fractional integral (5.18) as a function of t on $[0,1]$ for three different values of $\alpha = 1/4, 1/2$, and 1, are presented in Figure 5 (left).

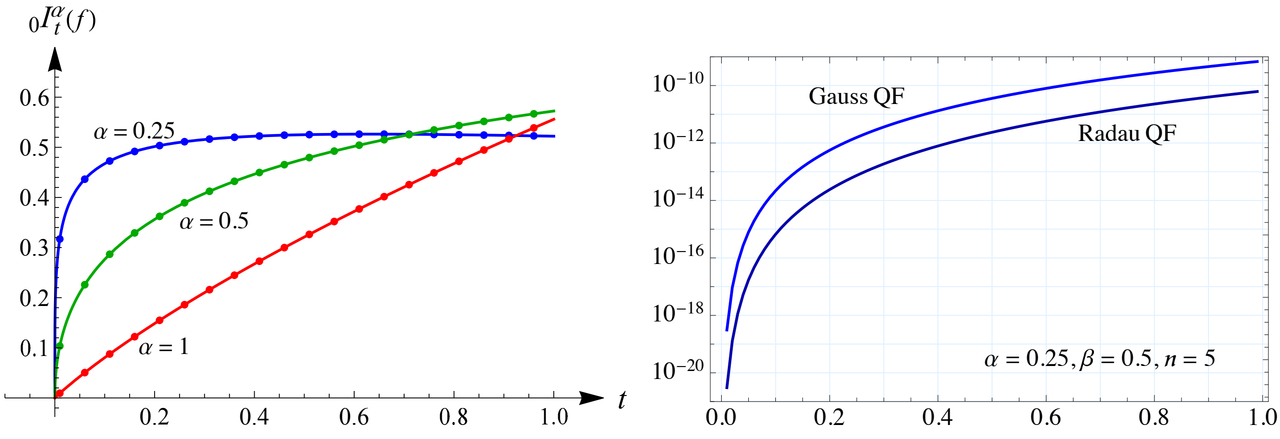

Figure 5. (Left) The fractional integral ${}_0I_t^{\alpha}(f)$ for $f(t) = \mathrm{e}^t\,\mathrm{erfc}(\sqrt{t})$, when $\alpha = 1/4, 1/2$, and 1; (Right) Relative errors in approximation by the Gauss and the Radau quadrature formula (in log-scale), for $\alpha = 1/4$, $\beta = 1/2$ and $n = 5$ nodes, when $0 \le t \le \pi/2$

Now, we apply the Gaussian quadrature formula (5.12) (from Theorem 5.1) for numerical calculating ${}_0I_t^{\alpha}(f)$.
From the series expansion of the function $z \mapsto f(z)$, given by (5.17), at the origin $z = 0$,

$$f(z) = 1 - \frac{2\sqrt{z}}{\sqrt{\pi}} + z - \frac{4z^{3/2}}{3\sqrt{\pi}} + \frac{z^2}{2} - \frac{8z^{5/2}}{15\sqrt{\pi}} + \frac{z^3}{6} + O\big(z^{7/2}\big),$$

we conclude that there is a critical singularity at $z = 0$ of these function, and it can slow down the convergence of the quadrature process (5.12), because this singularity is appeared also at $x = 0$ of the function $x \mapsto F_0(x; t, \alpha, \beta)$ $(a = 0)$, defined by (5.6), except certain cases when β takes some special values.

The corresponding series expansion in x of the function F_a is given by

$$\begin{aligned} x \mapsto F_0(x; t, \alpha, \beta) &= t^{\alpha} f\big(t x^{1/\beta}\big) = t^{\alpha} \mathrm{e}^{t x^{1/\beta}} \mathrm{erfc}\Big(\sqrt{t\, x^{1/\beta}}\Big) \\ &= t^{\alpha} \left(1 - 2\sqrt{\frac{t}{\pi}}\, x^{1/(2\beta)} + t\, x^{2/(2\beta)} - \frac{4t}{3}\sqrt{\frac{t}{\pi}}\, x^{3/(2\beta)} + \frac{t^2}{2} x^{4/(2\beta)} - \frac{8t^2}{15}\sqrt{\frac{t}{\pi}}\, x^{5/(2\beta)} + O\big(x^{6/(2\beta)}\big)\right). \end{aligned}$$

Putting $2\beta = 1/m$, where $m \in \mathbb{N}$, we obtain the following series expansion of (5.6) in x (free of singularity)

$$x \mapsto F_0\Big(x; t, \alpha, \frac{1}{2m}\Big) = t^\alpha \left(1 - 2\sqrt{\frac{t}{\pi}}\, x^m + t\, x^{2m} - \frac{4t}{3}\sqrt{\frac{t}{\pi}}\, x^{3m} + \frac{t^2}{2}\, x^{4m} - \frac{8t^2}{15}\sqrt{\frac{t}{\pi}}\, x^{5m} + O(x^{6m})\right). \tag{5.19}$$

This requires the weight function

$$x \mapsto w\Big(x; \alpha, \frac{1}{2m}\Big) = \frac{2m}{\Gamma(\alpha)} x^{2m-1}(1 - x^{2m})^{a-1} \quad (m \in \mathbb{N}),$$

whose moments are given by

$$\mu_k^{(m)} = \int_0^1 x^k w\Big(x; \alpha, \frac{1}{2m}\Big)\, \mathrm{d}x = \frac{\Gamma\left(\frac{k}{2m} + 1\right)}{\Gamma\left(a + \frac{k}{2m} + 1\right)}, \quad k = 0, 1, \dots .$$

The simplest case is for $m = 1$, i.e., when $\beta = 1/2$, and then the weight function

$$x \mapsto w(x; \alpha, 1/2) = \frac{2}{\Gamma(\alpha)} x(1 - x^2)^{a-1},$$

is the simplest, whose moments are given by

$$\mu_k = \mu_k^{(1)} = \int_0^1 x^k w\big(x; \alpha, \tfrac{1}{2}\big)\, \mathrm{d}x = \frac{\Gamma\left(\frac{k}{2} + 1\right)}{\Gamma\left(a + \frac{k}{2} + 1\right)}, \quad k = 0, 1, \dots .$$

The case $\alpha = \beta = 1/2$ is considered in Subsection 4.1.

For this choice of $\beta\,(= 1/2)$ the convergence of the quadrature process (5.12) (Theorem 5.1) is very fast. Taking only $n = 5$ nodes in the quadrature sum $Q_n(F_0(\,\cdot\,; t_\nu, \alpha, 1/2); w)$ $(t_\nu = \nu/100,\ \nu = 0, 1, \dots, 100)$, for $\alpha = 1/4, 1/2$ and 1, we obtain numerical values of ${}_0I_t^\alpha f(t_\nu)$. Each fifth point in the corresponding graphics in Figure 5 (left) is displayed, and show a good match with the exact values. Interpolation curves for the relative errors (5.16) of all evaluated points for $\alpha = 1/4$, i.e., $t \mapsto \mathrm{Err}_5^{(1/4,1/2)} f(t)$, $0 \le t \le 1$, is presented in Figure 5 (right). However, an application of the Radau quadrature rule (5.14)

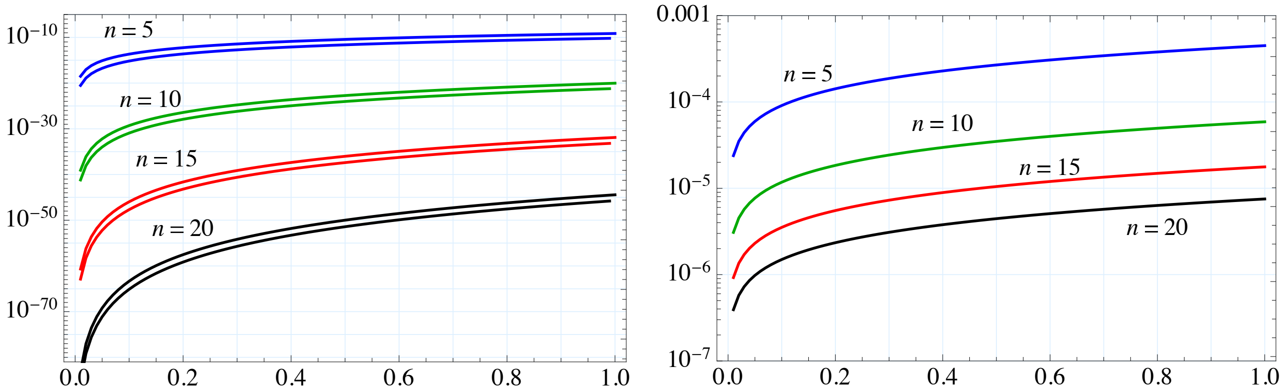

Figure 6. Relative errors $t \mapsto \mathrm{Err}_n^{(1/4,1/2)} f(t)$ (in log-scale), obtained by Gauss-Christoffel and Radau rules for $n = 5, 10, 15$, and 20 nodes (left); Relative errors $t \mapsto \mathrm{Err}_n^{(1/2,1)} f(t)$ (in log-scale) obtained by the Gauss-Christoffel rule for $n = 5, 10, 15, 20$ (right)

(Theorem 5.2), with also $n = 5$ points, gives better approximation $t \mapsto Q_5^{(1)}(F_0(\,\cdot\,; t, 1/4, 1/2); w)$ of the considered fractional integral and it is presented in the same figure. Both of these graphics for relative errors $t \mapsto \mathrm{Err}_n^{(1/4,1/2)} f(t)$, obtained by the Gauss-Christoffel rule $Q_n(F_0(\,\cdot\,; t, 1/4, 1/2); w)$ and the Radau rule $Q_n^{(1)}(F_0(\,\cdot\,; t, 1/4, 1/2); w)$, are presented in Figure 6 (left) for $n = 5, 10, 15$, and 20 nodes. By comparing the obtained results, we can conclude that for each number of nodes n, the Radau rule gives a more accurate approximation for about two orders of magnitude in relation to the Gaussian approximation. Very similar situation is for other values of α.

However, if we take the parameter $\beta \ne 1/2$, the convergence of quadrature sums $QS_n^{(\alpha,\beta)}[f; t]$, obtained by Gaussian and Radau rules, are significantly slower. Relative errors in the Gaussian approximation for $\alpha = 1/2$, $\beta = 1$, and $n = 5, 10, 15, 20$, are displayed in Figure 6 (right). Similarly, for $\beta = 1/3$ and $1/4$, the corresponding graphics are presented in Figure 7 for the same value of α.

As we can see, in the case when $\beta = 1/4$ $(m = 2)$ the convergence is faster, but not as in the previous case for $\beta = 1/2$ $(m = 1)$, when the n-point quadrature formula integrates exactly the first $2n$ terms in the expansion (5.19), i.e., all ones with degree at most $2n - 1$. However, for $m = 2$, such a formula integrates exactly only the first n terms in

$$x \mapsto F_0\Big(x; t, \alpha, \frac{1}{4}\Big) = t^\alpha f(tx^4) = t^a\left(1 - 2\sqrt{\frac{t}{\pi}}\, x^2 + tx^4 - \frac{4t}{3}\sqrt{\frac{t}{\pi}}\, x^6 + \frac{t^2}{2} x^8 - \frac{8t^2}{15}\sqrt{\frac{t}{\pi}}\, x^{10} + O(x^{12})\right),$$

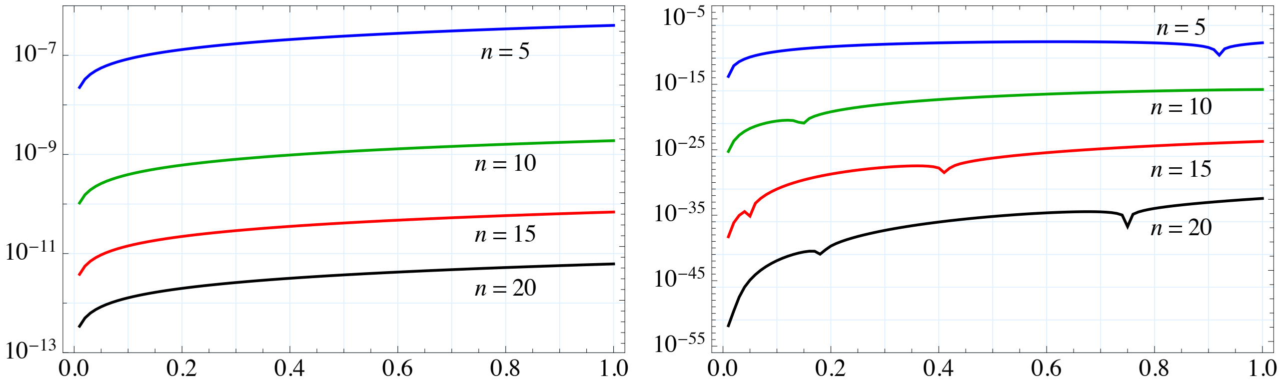

Figure 7. Relative errors $t \mapsto \mathrm{Err}_n^{(1/2,\beta)} f(t)$ (in log-scale), obtained by the Gauss-Christoffel rule with $n = 5, 10, 15, 20$ nodes, when the parameter $\beta = 1/3$ (left) and $\beta = 1/4$ (right)

because $0 \le 2\nu < 2n-1$ gives $0 \le \nu \le n-1$. This means that for $\beta = 1/4$ $(m = 2)$ the relative error in quadrature rule with $2n$ nodes is of the same order as the error with n nodes for $\beta = 1/2$ $(m = 1)$.

Example 5.2 Now we consider the right fractional Riemann-Liouville integral (5.5) with $b = \pi$ and $f(t) = \sin(t)$, i.e.,

$$ {}_tI_\pi^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^\pi (\tau - t)^{\alpha-1} \sin(\tau)\, d\tau, \quad t < \pi, $$

whose the exact value can be expressed in terms of hypergeometric function ${}_1F_2$ as

$$ {}_tI_\pi^\alpha f(t) = \frac{(\pi-t)^a}{\Gamma(a+2)} \left\{ (a+1)\sin(\pi t)\, {}_1F_2\left[\begin{matrix} \frac{a}{2} \\ \frac{1}{2}, \ \frac{a+2}{2} \end{matrix} \,\middle|\, -\frac{1}{4}\pi^2(\pi-t)^2 \right] + \pi a(\pi - t)\cos(\pi t)\, {}_1F_2\left[\begin{matrix} \frac{a+1}{2} \\ \frac{3}{2}, \ \frac{a+3}{2} \end{matrix} \,\middle|\, -\frac{1}{4}\pi^2(\pi-t)^2 \right] \right\}. $$

In order to apply our procedure to numerical calculation of this integral, according to (5.7), we use the function F_b defined by $x \mapsto F_b(x; t, \alpha) = (\pi - t)^\alpha \sin(\pi - (\pi - t)x) = (\pi - t)^\alpha \sin((\pi - t)x)$ and the weight function $x \mapsto w(x; \alpha, 1)$. In this case we expect fast convergence of the quadrature process because the function $x \mapsto F_b$ is entire for $\beta = 1$.

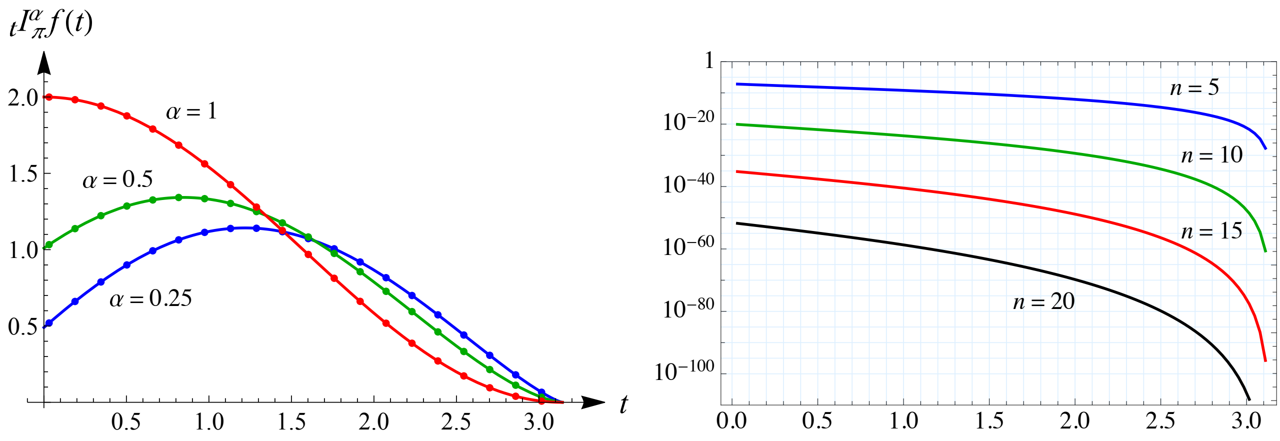

Figure 8. (Left) The fractional integral ${}_tI_\pi^\alpha(f)$ for $f(t) = \sin(t)$, when $\alpha = 1/4, 1/2$, and 1; (Right) Relative errors in quadrature approximation (in log-scale), for $\alpha = 1/4$, when number of quadrature nodes are $n = 5, 10, 15$, and 20

In Figure 8 (left) we present graphics of the exact values of the right fractional Riemann-Liouville integral ${}_tI_\pi^\alpha \sin(t)$ for $\alpha = 1/4, 1/2$, and 1, as well as values obtained numerically by quadrature formula (5.13) with $n = 5$ nodes. The corresponding relative errors (in log-scale) in the Gaussian quadrature sums (5.13) for $n = 5, 10, 15$ and 20 nodes are presented for $\alpha = 1/4$ in Figure 8 (right). The behaviour of relative errors for other values of α are very similar to the previous one. As we can see, the quadrature process converges very fast to ${}_tI_\pi^\alpha \sin(t)$ for each $0 \le t \le \pi$, in particular for larger t. For example, for only $n = 5$ nodes the relative error for $t = 0$ is 7.75×10^{-8}, i.e., the obtained result has at least seven exact decimal digits, while for t near π this number of exact digits is near 30. But, if we take $n = 20$ nodes, the relative error for $t = 0$ is even 2.56×10^{-52}.

An improvement can be obtained using the corresponding Radau quadrature (5.15) by adding a node at $x = 0$. The comparison with the Gaussian formula for $n = 5$ and $0 \le t \le \pi/2$ is presented in Figure 9 (left), when the Radau approximation gives at least nine decimal digits for $t < 0.1$ and more than ten digits for larger t. Comparisons for bigger number of nodes n are presented in the same figure (right). Note that the Radau modification (5.15) does not require a calculation in the added node $x = 0$, because for $b = \pi$, $F_\pi(0; t; \alpha) = (\pi - t)^\alpha f(\pi) = 0$.

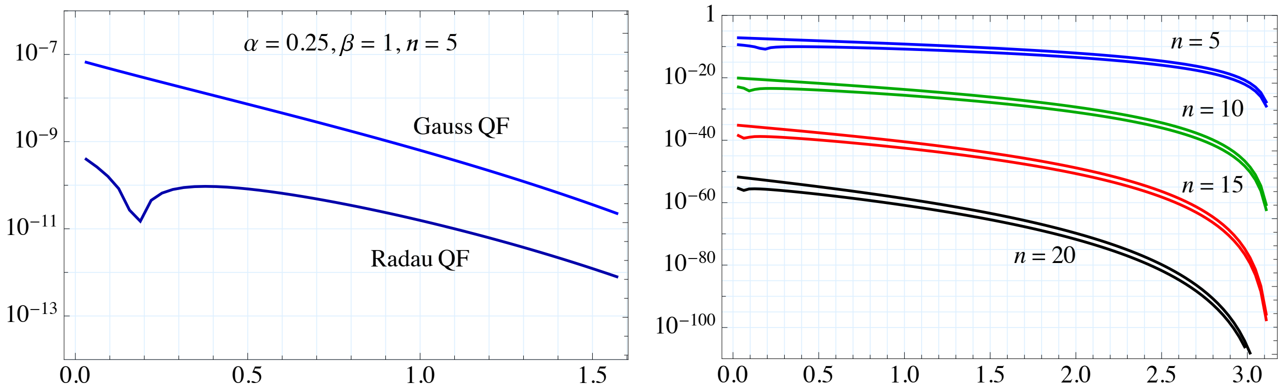

Figure 9. Relative errors in approximation by the Gauss and the Radau quadrature formula (in log-scale), for $\alpha = 1/4$, $\beta = 1$, and $n = 5$, $0 \le t \le \pi/2$ (left) and for $n = 5, 10, 15$, and 20, $0 \le t \le \pi$ (right)

Example 5.3 We take now $f(t) = \sin(\pi\sqrt{t})$, for which $\mathcal{L}[f(t)] = \frac{1}{2}(\pi/s)^{3/2}\mathrm{e}^{-\pi^2/(4s)}$, so that

$$ {}_0I_t^\alpha f(t) = \frac{\pi^{3/2}t^{\alpha+\frac{1}{2}}}{2\Gamma\left(a+\frac{3}{2}\right)}\,{}_0F_1\Big(;a+\frac{3}{2};-\frac{\pi^2 t}{4}\Big), \tag{5.20} $$

where ${}_0F_1$ is confluent hypergeometric function defined by

$$ {}_0F_1\left[\begin{array}{c|c} - & \\ b & z \end{array}\right] = \sum_{k=0}^{\infty}\frac{1}{(b)_k}\cdot\frac{z^k}{k!}. $$

This function is closely related to the Bessel function, so that (5.20) becomes

$$ {}_0I_t^\alpha f(t) = 2^{\alpha-\frac{1}{2}}\pi^{1-\alpha}t^{\frac{1}{2}(\alpha+\frac{1}{2})}J_{\alpha+\frac{1}{2}}\left(\pi\sqrt{t}\right), \tag{5.21} $$

and it is presented in Figure 10 (left) for $\alpha = 1/4, 1/2$, and 1.

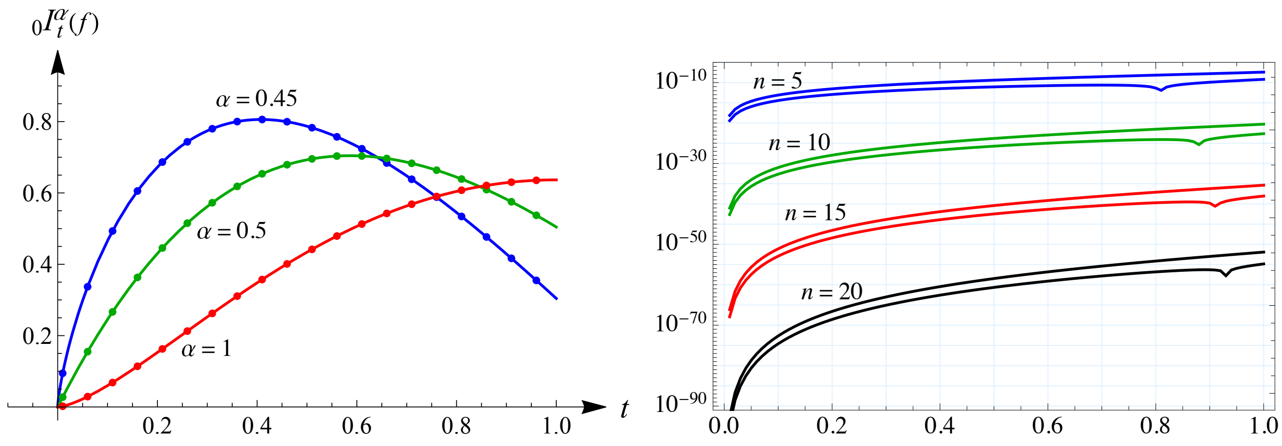

Figure 10. (Left) The fractional integral ${}_0I_t^\alpha(f)$ for $f(t) = \sin(\pi\sqrt{t})$, when $\alpha = 1/4, 1/2$, and 1; (Right) Relative errors in Gaussian and Radau quadrature approximations (in log-scale), for $\alpha = 1/4$, $\beta = 1/2$, when number of quadrature nodes are $n = 5, 10, 15$, and 20

According to the expansion

$$ \sin(\pi\sqrt{t}) = \sqrt{t}\Big(\pi - \frac{\pi^3 t}{6} + \frac{\pi^5 t^2}{120} + O\left(t^3\right)\Big), $$

we conclude that the function $x \mapsto F_0\big(x^{1/\beta}; t, \alpha\big) = t^\alpha f\big(tx^{1/\beta}\big)$ becomes an entire function if we take $\beta = 1/2$, $1/4$, etc., but the largest value of this parameter is most appropriate, because of facts analyzed in Example 5.1. In this case, the weight

Table 2. Relative errors in quadrature sums obtained by n-point Gauss-Christoffel (GC) and Radau (R) rules in some selected values of $t \in [0,1]$ for $\alpha = 1/4$ and $\beta = 1/2$

n	rule	$t=0.01$	$t=0.05$	$t=0.1$	$t=0.2$	$t=0.3$	$t=0.5$	$t=0.8$	$t=1.0$
5	GC	7.47(−19)	2.41(−15)	8.07(−14)	2.83(−12)	2.38(−11)	3.89(−10)	6.82(−9)	3.81(−8)
	R	3.71(−20)	1.15(−16)	3.64(−15)	1.13(−13)	8.13(−13)	8.53(−12)	4.28(−12)	6.20(−10)
10	GC	8.96(−42)	9.08(−35)	9.76(−32)	1.11(−28)	7.12(−27)	1.52(−24)	2.90(−22)	5.05(−21)
	R	2.24(−43)	2.18(−36)	2.23(−33)	2.25(−30)	1.26(−28)	1.86(−26)	7.96(−25)	2.28(−23)
15	GC	7.07(−67)	2.24(−56)	7.71(−52)	2.81(−47)	1.38(−44)	3.81(−41)	7.68(−38)	4.13(−36)
	R	1.18(−68)	3.59(−58)	1.18(−53)	3.83(−49)	1.65(−46)	3.22(−43)	1.90(−40)	8.55(−39)
20	GC	2.01(−93)	1.99(−79)	2.20(−73)	2.56(−67)	9.57(−64)	3.42(−59)	7.26(−55)	1.20(−52)
	R	2.51(−95)	2.40(−81)	2.52(−75)	2.64(−69)	8.67(−66)	2.21(−61)	1.53(−57)	1.42(−55)

function in (5.2) is also the simplest, i.e., $w(x;\alpha,1/2)$. Using this weight function and Gaussian quadrature (5.11) with only $n=5$ nodes we obtain approximative numerical values of ${}_0I_t^{\alpha}f(t_\nu)$, $\nu = 1,2,\ldots,100$. Each fifth point in the corresponding graphics in Figure 10 (left) is displayed, and show a good match with the exact values.

Interpolation curves for the relative errors (5.16) of all evaluated points for $\alpha = 1/4$, i.e., $t \mapsto \mathrm{Err}_n^{(1/4,1/2)}f(t)$ $(0 \le t \le 1)$ for $n = 5, 10, 15$ and 20 nodes in the quadrature sums obtained by Gauss-Christoffel rule (5.12) and Radau rule (5.14) are presented in Figure 10 (right). In Table 2 we give numerical values of the corresponding relative errors in some selected points of $t \in [0,1]$. Obviously, Radau's quadrature formula gives better results (with smaller relative errors), which we pointed out earlier.

6. Conclusion

In addition to the development of several classes of orthogonal polynomials and corresponding Gaussian-type quadrature formulas with specially selected weight functions on a finite interval, in this the paper we present two efficient methods for numerical calculation of the left and right fractional Riemann-Liouville integrals. By a few examples we illustrate the efficiency of the proposed numerical procedures.

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