New fixed point results in extended $b-$metric-like spaces via simulation functions with applications

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Abstract

The main ambition proposed in this article is to provide new fixed point results for triangular $\alpha-$orbital admissible contractions via some auxiliary and simulation functions in the frame of extended $b-$metric-like spaces. As an application, we prove the existence of a unique solution for a nonlinear fractional differential equation with exponential weighted integral boundary conditions via the generalized proportional fractional derivative of Caputo type with order $\beta \in (n-1, n]$. Further, we demonstrate the usability of our results through several examples.

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1. Introduction

During the last decades, many researchers have focused on joining fixed point theory with fractional calculus that deals with integrals and derivatives of arbitrary orders. Fractional differential equations help model real-life problems into mathematical models more accurately than the traditional ones, and fixed point theorems can provide unique solutions for these equations.

Since, the outstanding fixed point result of Banach has been extended by using different forms of contractive conditions in various generalized metric spaces. Some of the most important generalizations of metric space are $b-$metric space in [9, 12], partial metric space in [28], partial $b-$metric space in [36], metric-like space in [6], $b-$metric-like space in [4, 16].

Recently, a concept which is an extension of $b-$metric was presented by Parvaneh and Ghoncheh [30, 31] under the name extended $b-$metric or $p-$metric. In consequence, the authors in [32] proposed the concept of partial $p-$metric. Further, they generalized all of the above mentioned spaces by introducing the notion of $p-$metric-like in [33] and presented some basic properties of such spaces with some fixed point results for JSHR-contractions in the setup of such spaces. Khojasteh et al. [25] introduced the notion of a simulation function and consider a new class of contractions called $\mathcal{Z}-$contractions with a view to unify several existing fixed point results in the literature. In 2019, Karapinar et al. [18] considered different families of auxiliary functions in order to prove some fixed point results for a variety of triangular $\alpha-$orbital admissible contractions defined on complete metric spaces. Further, they applied their results on the existence of solutions of both ordinary and fractional boundary value problems.

On the other hand, it turned out that the methods used in the fractional calculus are splendid when modeling long-memory processes and many phenomena that appear in physics, chemistry, electricity, mechanics and many other disciplines [13, 14, 21, 26, 27, 29]. In recent years, several researchers in the field of fractional calculus felt the need for other types of fractional operators to be more suitable for more complex systems. In [1, 20], the authors proposed the conformable derivative which lacks the essential property that the zero-order derivative for a function must be yield the function itself. In order to circumvent this deficit, the authors in [7] redefined the conformable derivative. Following this way, in [5, 17] the fractional version of the modified conformable derivative was suggested.

Following this tendency, fixed point results presented in [18] were extended to be considered on $p-$metric-like spaces. The concept of extension was based on using simulation function and relaxing some conditions imposed on the considered auxiliary functions. Then we apply the obtained fixed point results to provide sufficient conditions for the existence of a unique solution for the following nonlinear fractional differential equation.

$$\begin{aligned} &{}_0^cD^{\beta,\rho}x(t) = f\big(t, x(t)\big),\ \ t \in J = [0,T],\ n-1<\beta\le n,\ \rho\in(0,1],\\ &x^{(i)}(t)|_{t=0} = 0, x(1) = \int_0^r e^{\lambda s}x(s)ds,\ \ 0\le i\le n-2,\ r\in(0,1),\ \lambda = \frac{1-\rho}{\rho}, \end{aligned} \tag{1.1}$$

where $x : J \to \mathbb{R}$, $f : J\times\mathbb{R}\to\mathbb{R}$ are continuous functions and ${}_0^cD^{\beta,\rho}$ denotes the generalized proportional fractional derivative of Caputo type of order $\beta > 0$.

2. **Basic Concepts**

We begin with giving some notations and preliminaries related with $p-$metric, partial $p-$metric, $p-$metric-like spaces, simulation function and some auxiliary functions needed to state our results. Let $\mathbb{R}^+$ ($\mathbb{N}_0$) be the set of nonnegative reals (integers) and Υ be given as:

$$\big\{\Omega : \mathbb{R}^+ \to \mathbb{R}^+ : \Omega \text{ is a strictly increasing continuous function satisfying } \Omega^{-1}(t) \le t \le \Omega(t)\big\}.$$

Definition 2.1. [31] *Let X be a nonempty set and $\Omega \in \Upsilon$. A function $d_p : X\times X \to \mathbb{R}^+$ is a $p-$metric if it satisfies the following conditions for all $x, y, z \in X$:*

(d_{p_1}) $d_p(x,y) = 0 \Leftrightarrow x = y$,
(d_{p_2}) $d_p(x,y) = d_p(y,x)$,
(d_{p_3}) $d_p(x,y) \le \Omega\big(d_p(x,z) + d_p(z,y)\big)$.

The pair (X, d_p) is called a $p-$metric space.

Remark 2.1. *Every metric is a $p-$metric with $\Omega(t) = t$ and every $b-$metric with parameter $s \ge 1$ is a $p-$metric with $\Omega(t) = st$ but not vice versa (see the next example).*

Example 2.1. *Let (X, d) be any metric space and $\rho(x,y) = d(x,y)\cosh d(x,y)$, $\forall\, x, y \in X$, then (X,ρ) is a $p-$metric space with $\Omega(t) = t\cosh t$. For $x, y, z \in X$, we have*

$$\rho(x,y)=0 \Leftrightarrow d(x,y)=0 \Leftrightarrow x=y,$$
$$\rho(x,y)=\rho(y,x).$$

Using the fact that $\cosh t \geq 1, \ \forall\, t \in \mathbb{R}^+$, implies

$$\begin{aligned}\rho(x,y) &\leq [d(x,z)+d(z,y)]\cosh[d(x,z)+d(z,y)]\\ &\leq [\rho(x,z)+\rho(z,y)]\cosh[\rho(x,z)+\rho(z,y)]\\ &\leq \Omega\big(\rho(x,z)+\rho(z,y)\big).\end{aligned}$$

Further, if we define $X=[0,1]$ and $d(x,y)=|x-y|$, then (X,ρ) is not a metric space for

$$\rho(0,1)=1.5 \geq 2\times 0.56=\rho(0,0.5)+\rho(0.5,1).$$

Definition 2.2. [32] *Let X be a nonempty set and $\Omega \in \Upsilon$. A function $p_p : X\times X \to \mathbb{R}^+$ is a partial $p-$metric if for any $x,y,z \in X$, the following conditions are satisfied:*

(p_{p_1}) $p_p(x,y)=p_p(x,x)=p_p(y,y) \ \Leftrightarrow \ x=y,$
(p_{p_2}) $p_p(x,x)\leq p_p(x,y),$
(p_{p_2}) $p_p(x,y)=p_p(y,x),$
(p_{p_3}) $p_p(x,y)-p_p(x,x)\leq \Omega\big(p_p(x,z)+p_p(z,y)-p_p(z,z)-p_p(x,x)\big).$

The pair (X,p_p) is called a partial $p-$metric space.

Remark 2.2. *If we define $\Omega(t)=t$, then a partial $p-$metric becomes partial metric and If $\Omega(t)=st$ for $s\geq 1$, then a partial $p-$metric becomes partial $b-$metric with parameter s.*

Definition 2.3. [33] *Let X be a nonempty set and $\Omega \in \Upsilon$. A function $\sigma_p : X\times X\to\mathbb{R}^+$ is a $p-$metric-like if for any $x,y,z\in X$, the following conditions hold:*

(σ_{p_1}) $\sigma_p(x,y)=0 \ \Rightarrow \ x=y,$
(σ_{p_2}) $\sigma_p(x,y)=\sigma_p(y,x),$
(σ_{p_3}) $\sigma_p(x,y)\leq \Omega\big(\sigma_p(x,z)+\sigma_p(z,y)\big).$

The pair (X,σ_p) is called a $p-$metric-like space.

Remark 2.3. *Since, every partial metric is a metric-like and every metric-like is a $p-$metric-like with $\Omega(t)=t$. Every $b-$metric-like with parameter $s\geq 1$ is a $p-$metric like with $\Omega(t)=st$. Also, every partial $p-$metric is a $p-$metric-like with a super additive function $\Omega\in\Upsilon$. However, the reverse implications do not hold in general.*

Proposition 2.1. *Let (X,σ_b) be a $b-$metric-like space with coefficient $s\geq 1$ and $\rho(x,y)=\xi\big(\sigma_b(x,y)\big)$ where $\xi :\to \mathbb{R}^+$ is a strictly increasing function with $t\leq \xi(t)$, $\forall\, t$ and $0=\xi(0)$. Then (X,ρ) is a $p-$metric-like space with $\Omega(t)=\xi(st)$.*

Proof. For each $x,y,z\in X$, we have

$$\rho(x,y)=0\Rightarrow \sigma_b(x,y)=0\Rightarrow x=y,$$
$$\rho(x,y)=\rho(y,x)$$

and

$$\begin{aligned}\rho(x,y) &= \xi\big(\sigma_b(x,y)\big)\\ &\le \xi\big(s[\sigma_b(x,z)+\sigma_b(z,y)]\big)\\ &\le \xi\big(s[\xi(\sigma_b(x,z))+\xi(\sigma_b(z,y))]\big)\\ &\le \xi\big(s[\rho(x,z)+\rho(z,y)]\big)\\ &\le \Omega\big(\rho(x,z)+\rho(z,y)\big).\end{aligned}$$

Hence, ρ is a $p-$metric-like on X. □

The aforesaid proposition provides several examples on a $p-$metric-like space.

Example 2.2. *Let (X,σ_b) be a $b-$metric-like space with coefficient $s\ge 1$, then*

1. $\rho(x,y)=e^{\sigma_b(x,y)}-1$ *is a $p-$metric-like with* $\Omega(t)=e^{st}-1$.
2. $\rho(x,y)=\sigma_b(x,y)+\ln\big(1+\sigma_b(x,y)\big)$ *is a $p-$metric-like with* $\Omega(t)=st+\ln(1+st)$.
3. $\rho(x,y)=\sigma_b(x,y)\cosh\big(\sigma_b(x,y)\big)$ *is a $p-$metric-like with* $\Omega(t)=st\cosh(st)$.
4. $\rho(x,y)=\big(\sigma_b(x,y)\big)^q+\sigma_b(x,y)$ *is a $p-$metric-like with* $\Omega(t)=(st)^q+st$, $q\in\mathbb{N}$.
5. $\rho(x,y)=e^{\sigma_b(x,y)}\ln\big(1+\sigma_b(x,y)\big)$ *is a $p-$metric-like with* $\Omega(t)=e^{st}\ln(1+st)$.
6. $\rho(x,y)=e^{\sigma_b(x,y)}\sinh^{-1}(e^{\sigma_b(x,y)})$ *is a $p-$metric-like with* $\Omega(t)=e^{st}\sinh^{-1}(e^{st})$.
7. $\rho(x,y)=e^{\sigma_b(x,y)}\sec^{-1}(e^{\sigma_b(x,y)})$ *is a $p-$metric-like with* $\Omega(t)=e^{st}\sec^{-1}(e^{st})$.
8. $\rho(x,y)=e^{\sigma_b(x,y)}\tan^{-1}(e^{\sigma_b(x,y)}-1)$ *is a $p-$metric-like with* $\Omega(t)=e^{st}\tan^{-1}(e^{st}-1)$.

Every $p-$metric-like σ_p on a nonempty set X generates a topology τ_{σ_p} on X whose base is the set of all open balls $B(x,\epsilon)$, $\forall\ x\in X$ and $\epsilon>0$, where

$$B(x,\epsilon)=\{y\in X : |\sigma_p(x,y)-\sigma_p(x,x)|<\epsilon\}.$$

Definition 2.4. [33] *Let (X,σ_p) be a $p-$metric-like space, $x\in X$ and $\{x_n\}$ be a sequence in X. Then,*

(i) $\{x_n\}$ *is said to converge to x with respect to τ_{σ_p} (we may write, $x_n\longrightarrow x$), if*

$$\lim_{n\to\infty}\sigma_p(x_n,x)=\sigma_p(x,x).$$

(ii) $\{x_n\}$ *is said to be Cauchy in (X,σ_p), if*

$$\lim_{n,m\to\infty}\sigma_p(x_n,x_m)$$

exists and is finite.

(iii) (X,σ_p) *is said to be complete, if for every Cauchy sequence $\{x_n\}$ in X, there exists $x\in X$ such that*

$$\lim_{n,m\to\infty}\sigma_p(x_n,x_m)=\lim_{n\to\infty}\sigma_p(x_n,x)=\sigma_p(x,x).$$

Lemma 2.1. *Let (X,σ_p) be a $p-$metric-like space with parameter $s\ge 1$ and $\{x_n\}$ be a convergent sequence in X such that*

$$\lim_{n,m\to\infty}\sigma_p(x_n,x_m)=\lim_{n\to\infty}\sigma_b(x_n,x)=\sigma_b(x,x)=0,\quad x\in X. \tag{2.1}$$

Then, every subsequence $\{x_{n_k}\}$ with $n_k\ge k\in\mathbb{N}$ converges to the same limit $x\in X$.

Proof. Since, $x_n \to x$ and $\lim_{n\to\infty} \sigma_p(x_n, x) = 0$, then for a given $\epsilon > 0$

$$\exists \ n_0 \in \mathbb{N} \ : \ n > n_0 \ \Rightarrow \ \sigma_p(x_n, x) < \epsilon.$$

From (σ_{p_3}) and (2.1), we have

$$\sigma_p(x_{n_k}, x) < \Omega\big[\sigma_p(x_{n_k}, x_k) + \sigma_p(x_k, x)\big] \to 0 \text{ as } n_k \geq k \to \infty.$$

Therefore,

$$\lim_{k\to\infty} \sigma_p(x_{n_k}, x) = 0 = \sigma_p(x, x). \tag{2.2}$$

□

Lemma 2.2. [33] *Let (X, σ_p) be a $p-$metric-like space and $\{x_n\}$ be a convergent sequence to a point x in X. Then, for any $y \in X$, we have*

$$\begin{aligned}\Omega^{-1}\big[\sigma_p(x, y)\big] - \sigma_p(x, x) &\leq \liminf_{n\to\infty} \sigma_p(x_n, y)\\ &\leq \limsup_{n\to\infty} \sigma_p(x_n, y) \leq \Omega\big[\sigma_p(x, y) + \sigma_p(x, x)\big].\end{aligned}$$

The notions of (triangular) $\alpha-$admissible mappings were defined at the first time in [19, 35] and then these notions were modified in [34].

Definition 2.5. [34] *Let (X, σ_p) be a $p-$metric-like space, T be a self-mapping on X and $\alpha : X \times X \to \mathbb{R}^+$ be a function, then*

(1) *T is called $\alpha-$orbital admissible* [34], *if*

$$x \in X, \ \alpha(x, Tx) \geq 1 \ \ \Rightarrow \ \ \alpha(Tx, T^2x) \geq 1.$$

(2) *T is called triangular $\alpha-$orbital admissible* [34], *if T is $\alpha-$orbital admissible and*

$$x, y \in X, \ \alpha(x, y) \geq 1 \text{ and } \alpha(y, Ty) \geq 1 \ \ \Rightarrow \ \ \alpha(x, Ty) \geq 1.$$

(3) *(X, σ_p) is called $\alpha-$complete* [15] *if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1, \ \forall \ n \in \mathbb{N}$, converges in the same manner shown in Definition 2.4.*

(4) *(X, σ_p) is called $\alpha-$regular* [11], *if for a point x and a sequence $\{x_n\}$ in X*

$$x_n \longrightarrow x \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \Rightarrow \alpha(x_n, x) \geq 1, \ \forall \ n \in \mathbb{N}.$$

Definition 2.6. [18] *Let (X, d) be a metric space and $\mathcal{A}$ be the family of auxiliary functions $h : X \times X \to \mathbb{R}^+$ such that*

$$\lim_{n\to\infty} h(x_n, y_n) = 1 \ \Rightarrow \ \lim_{n\to\infty} d(x_n, y_n) = 0. \tag{2.3}$$

Definition 2.7. [24] *Let Ψ be the set of all altering distance functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy:*

(1) *ψ is continuous and strictly increasing,*
(2) *$\psi(t) = 0 \ \Leftrightarrow \ t = 0$.*

Definition 2.8. [25] *Let $\mathcal{Z}$ be the set of all simulation functions $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ that satisfy:*

(ζ_1) *$\zeta(t, s) < s - t, \ \forall \ t, s > 0$,*
(ζ_2) *If $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$, then*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0 \Rightarrow \limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Definition 2.9. *Consider $0 < \rho \leq 1$, $\beta \in \mathbb{C}$ and $n = [Re(\beta)] + 1$.*

1. *The proportional derivative of order ρ is defined by*
$$D^{\rho} f(t) = (1-\rho) f(t) + \rho \acute{f}(t). \tag{2.4}$$
It is easy to figure out that
$$\lim_{\rho \to 0^+} D^{\rho} f(t) = f(t) \ \ \text{and} \ \ \lim_{\rho \to 1^-} D^{\rho} f(t) = \acute{f}(t).$$
*Thus, the derivative given in (**??**) is somehow considered to be more general than the conformable derivative which evidently does not tend to the original function as ρ tends to 0. Furthermore, $D^{n,\rho}$ can be defined by*
$$\begin{aligned} D^{n,\rho} f(t) &= \underbrace{\left(D^{\rho} \ldots D^{\rho}\right)}_{n-times} f(t) \\ &= \left((1-\rho) + \rho \frac{d}{dt}\right)^n f(t). \end{aligned}$$
2. *The generalized proportional fractional (GPF) integral of order β ($Re(\beta) > 0$) starting from a has the from*
$$\begin{aligned} {}_aI^{\beta,\rho} f(t) &= \frac{1}{\rho^{\beta}\Gamma(\beta)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{\beta-1} f(s) ds \\ &= \rho^{-\beta} e^{\frac{\rho-1}{\rho}t} \big({}_aI^{\beta}\big(e^{\frac{1-\rho}{\rho}t} f(t)\big)\big). \end{aligned}$$
If we let $\rho = 1$, then one can obtain the definition of Riemann-Liouville fractional derivative ${}_aI^{\beta}$.
3. *The GPF derivative of order β ($Re(\beta) \geq 0$) is given as*
$$\begin{aligned} {}_aD^{\beta,\rho} f(t) &= D^{n,\rho} {}_aI^{n-\beta,\rho} f(t) \\ &= \frac{D^{n,\rho}}{\rho^{n-\beta}\Gamma(n-\beta)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{n-\beta-1} f(s) ds. \end{aligned}$$
It is obvious that
$$\lim_{\beta \to 0} {}_aD^{\beta,\rho} f(t) = f(t) \ \ \text{and} \ \ \lim_{\beta \to 1} {}_aD^{\beta,\rho} f(t) = D^{\rho}(t).$$
4. *The GPF derivative of Caputo type of order β ($Re(\beta) \geq 0$) becomes*
$$\begin{aligned} {}_a^CD^{\beta,\rho} f(t) &= \big({}_aI^{n-\beta,\rho} D^{n,\rho} f\big)(t) \\ &= \frac{1}{\rho^{n-\beta}\Gamma(n-\beta)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{n-\beta-1} \big(D^{n,\rho} f\big)(s) ds. \end{aligned}$$

Lemma 2.3. *For $\rho \in (0,1]$ and $n = [Re(\beta)] + 1$, we have*
$${}_aI^{\beta,\rho} \ {}_a^cD^{\beta,\rho} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(D^{k,\rho} f)(a)}{\rho^k \Gamma(k+1)} (t-a)^k e^{\frac{\rho-1}{\rho}(t-a)}.$$

Lemma 2.4. *Let f be integrable on $t > a$ and $Re(\beta) > 0$, $\rho > 0$, $n = [Re(\beta)] + 1$. Then we have*
$${}_a^cD^{\beta,\rho} \ {}_aI^{\beta,\rho} f(t) = f(t).$$

Proposition 2.2. *Let $\alpha, \beta \in \mathbb{C}$ be such that $Re(\beta) \geq 0$. Then, for any $\rho > 0$, we have*

(1) $\big({}_aI^{\alpha,\rho} e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1}\big)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)\rho^{\alpha}} e^{\frac{\rho-1}{\rho}x}(x-a)^{\alpha+\beta-1},\ Re(\alpha) > 0.$

(2) $\big({}_aD^{\alpha,\rho} e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1}\big)(x) = \frac{\rho^{\alpha}\Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{\frac{\rho-1}{\rho}x}(x-a)^{\beta-1-\alpha},\ Re(\alpha) \geq 0.$

Proposition 2.3. *For $\beta \in \mathbb{C}$ with $Re(\beta) > 0$ and $\rho \in (0,1]$, $n = [Re(\beta)] + 1$, we have*

$$ {}_a^cD^{\beta,\rho} f(t) = {}_aD^{\beta,\rho} f(t) - \sum_{k=0}^{n-1} \frac{\rho^{\beta-k}(D^{k,\rho}f)(a)}{\Gamma(k+1-\beta)}(t-a)^{k-\beta} e^{\frac{\rho-1}{\rho}(t-a)}. $$

3. Fixed Point Results

Theorem 3.1. *Let (X, σ_p) be a $p-$metric-like space and $T : X \to X$ be a triangular $\alpha-$orbital admissible mapping. Suppose that for all $x, y \in X$ with $\alpha(x, y) \geq 1$,*

$$ \zeta\big(\alpha(x,y)\psi(\Omega(\sigma_p(Tx,Ty))), h(x,y)\psi(\sigma_p(x,y))\big) \geq 0,\ \ \zeta \in \mathcal{Z},\ \ \psi \in \Psi, \tag{3.1} $$

where

$$ R(x,y) = \max\left\{ \frac{\sigma_p(x,Tx)\sigma_p(y,Ty)}{\sigma_p(x,y)}, \sigma_p(x,y), \sigma_p(x,Tx), \right. $$
$$ \left. \sigma_p(y,Ty), \frac{\Omega^{-1}}{2}\left[\frac{\sigma_p(x,Ty)+\sigma_p(y,Tx)}{2}\right] \right\}. $$

Consider that the following properties hold true:

(a) *(X, σ_p) is $\alpha-$complete and $\alpha-$regular.*

(b) *For all $x, y \in Fix(T)$, we have $\alpha(x, y) \geq 1$, where $Fix(T)$ denotes the set of fixed points of T.*

Moreover, if there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a unique fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$ and define a sequence $\{x_n\} \subset X$ as

$$ x_{n+1} = Tx_n,\ \forall\, n \in \mathbb{N}_0. \tag{3.2} $$

Regarding that T is $\alpha-$ orbital admissible, we deduce by induction that

$$ \alpha(x_n, x_{n+1}) \geq 1,\ \forall\, n. \tag{3.3} $$

If $\sigma_b(x_n, x_{n+1}) = 0$ for some n, then $x_n = x_{n+1}$ and hence x_n is a fixed point of T. So, we assume that

$$ \sigma_b(x_n, x_{n+1}) > 0,\ \forall\, n. \tag{3.4} $$

From (3.2) - (3.4), we apply (3.1) at $x = x_{n-1}$ and $y = x_n$ to get

$$ \begin{aligned} 0 &\leq \zeta\big(\alpha(x_{n-1},x_n)\psi(\Omega(\sigma_p(x_n,x_{n+1}))), h(x_{n-1},x_n)\psi(R(x_{n-1},x_n))\big) \\ &< h(x_{n-1},x_n)\psi(R(x_{n-1},x_n)) - \alpha(x_{n-1},x_n)\psi(\Omega(\sigma_p(x_n,x_{n+1}))), \end{aligned} \tag{3.5} $$

where

$$ R(x_{n-1},x_n) = \max\left\{ \frac{\sigma_p(x_{n-1},x_n)\sigma_p(x_n,x_{n+1})}{\sigma_p(x_{n-1},x_n)}, \sigma_p(x_{n-1},x_n), \sigma_p(x_{n-1},x_n), \right. $$
$$ \left. \sigma_p(x_n,x_{n+1}), \frac{\Omega^{-1}}{2}\left[\frac{\sigma_p(x_{n-1},x_{n+1})+\sigma_p(x_n,x_n)}{2}\right] \right\}. $$

If $\sigma_p(x_{n-1}, x_n) \le \sigma_p(x_n, x_{n+1})$ for some n, then

$$\begin{aligned}
&\frac{\Omega^{-1}}{2}\left[\frac{\sigma_p(x_{n-1}, x_{n+1}) + \sigma_p(x_n, x_n)}{2}\right] \\
&\qquad \le \frac{\Omega^{-1}}{2}\left[\frac{\Omega\big(\sigma_p(x_{n-1}, x_n) + \sigma_p(x_n, x_{n+1})\big) + \Omega\big(\sigma_p(x_n, x_{n+1}) + \sigma_p(x_{n+1}, x_n)\big)}{2}\right] \\
&\qquad \le \sigma_p(x_n, x_{n+1}), \\
&\quad \Longrightarrow \ R(x_{n-1}, x_n) = \sigma_p(x_n, x_{n+1}).
\end{aligned}$$

From (3.5), we get

$$\begin{aligned}
\psi(\sigma_p(x_n, x_{n+1})) &\le \alpha(x_{n-1}, x_n)\psi(\Omega(\sigma_p(x_n, x_{n+1}))) \\
&\qquad < h(x_{n-1}, x_n)\psi(\sigma_p(x_n, x_{n+1})) \le \psi(\sigma_p(x_n, x_{n+1})),
\end{aligned}$$

which is a contradiction. Hence,

$$\sigma_p(x_n, x_{n+1}) \le \sigma_p(x_{n-1}, x_n), \ \forall\, n. \tag{3.6}$$

Now, $\sigma_b(x_n, x_{n+1})$ is monotone decreasing sequence of positive reals. Therefore, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} \sigma_p(x_n, x_{n+1}) = r. \tag{3.7}$$

We show that $r = 0$. Suppose, to the contrary, that $r > 0$. Using (3.5) - (3.7) and the properties on ψ, we obtain

$$\begin{aligned}
\psi(r) \longleftarrow \psi(\sigma_p(x_n, x_{n+1})) &\le \alpha(x_{n-1}, x_n)\psi(\Omega(\sigma_p(x_n, x_{n+1}))) \\
&< h(x_{n-1}, x_n)\psi(\sigma_p(x_{n-1}, x_n)) \le \psi(\sigma_p(x_{n-1}, x_n)) \longrightarrow \psi(r) > 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \alpha(x_{n-1}, x_n)\psi(\Omega(\sigma_p(x_n, x_{n+1}))) = \lim_{n\to\infty} h(x_{n-1}, x_n)\psi(\sigma_p(x_{n-1}, x_n)) = \psi(r) > 0.$$

Now, we can apply (ζ_2) to obtain a contradiction, hence $r = 0$.

$$\lim_{n\to\infty} \sigma_p(x_n, x_{n+1}) = 0. \tag{3.8}$$

Now, we show that

$$\lim_{n,m\to\infty} \sigma_p(x_n, x_m) = 0. \tag{3.9}$$

Consider the sequence

$$R_k = \sup\big\{\sigma_p(x_n, x_m) : m \ge n \ge k\big\}, \ \forall\, k \in \mathbb{N}. \tag{3.10}$$

Moreover, one can figure out that

$$\lim_{k\to\infty} R_k = 0 \ \Rightarrow \lim_{n,m\to\infty} \sigma_p(x_n, x_m) = 0$$

and

$$0 \le \ldots \le R_{k+1} \le R_k \le \ldots \le R_1.$$

Hence, the sequence $\{R_k\}$ is decreasing and bounded below by zero. Consequently, there exists $r \ge 0$ such that

$$\lim_{k\to\infty} R_k = r. \tag{3.11}$$

In view of (3.10), we conclude that

$$\forall\, k \in \mathbb{N}\, (\frac{1}{k} > 0),\ \ \exists\, m_k \geq n_k \geq k\ :\ R_k - \frac{1}{k} < \sigma_p(x_{n_k}, x_{m_k}) < R_k.$$

Indeed, we use the nonzero terms of the sequence $\{\sigma_p(x_{n_k}, x_{m_k})\}_{k\in\mathbb{N}}$ to prove that $r = 0$ (we assume, on the contrary, that $r > 0$). Letting $k \to \infty$ in the above inequality, together with (3.11) imply

$$\lim_{k\to\infty} \sigma_b(x_{n_k}, x_{m_k}) = r. \tag{3.12}$$

Applying (3.1) with $x = x_{n_k-1}$, $y = x_{m_k-1}$ and using the facts that

$$\begin{aligned}\frac{\Omega^{-1}}{2}\left[\frac{\sigma_p(x_{n_k-1}, x_{m_k}) + \sigma_p(x_{m_k-1}, x_{n_k})}{2}\right] &\leq \frac{\Omega^{-1}}{2}\left[\Omega\big(\sigma_p(x_{n_k-1}, x_{m_k-1}) + \sigma_p(x_{n_k}, x_{m_k})\big)\right]\\ &\leq \frac{\sigma_p(x_{n_k-1}, x_{m_k-1}) + \sigma_p(x_{n_k}, x_{m_k})}{2}\\ &\leq \max\{\sigma_p(x_{n_k-1}, x_{m_k-1}), \sigma_p(x_{n_k}, x_{m_k})\}\end{aligned}$$

and

$$\sigma_p(x_{n_k-1}, x_{n_k}) \geq \sigma_p(x_{m_k-1}, x_{m_k}),\ \ (\text{for, } n_k \leq m_k),$$

together with the properties on ζ, ψ, α, h and T, we reach

$$\begin{aligned}0 &\leq \zeta\big(\alpha(x_{n_k-1}, x_{m_k-1})\psi(\Omega(\sigma_p(x_{n_k}, x_{m_k}))), h(x_{n_k-1}, x_{m_k-1})\psi(R(x_{n_k-1}, x_{m_k-1}))\big)\\ &< h(x_{n_k-1}, x_{m_k-1})\psi(R(x_{n_k-1}, x_{m_k-1})) - \alpha(x_{n_k-1}, x_{m_k-1})\psi(\Omega(\sigma_p(x_{n_k}, x_{m_k}))),\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}R(x_{n_k-1}, x_{m_k-1}) &= \max\left\{\frac{\sigma_p(x_{n_k-1}, x_{n_k})\sigma_p(x_{m_k-1}, x_{m_k})}{\sigma_p(x_{n_k-1}, x_{m_k-1})}, \sigma_p(x_{n_k-1}, x_{m_k-1}), \sigma_p(x_{n_k-1}, x_{n_k}),\right.\\ &\qquad\left.\sigma_p(x_{m_k-1}, x_{m_k}), \frac{\Omega^{-1}}{2}\left[\frac{\sigma_p(x_{n_k-1}, x_{m_k}) + \sigma_p(x_{m_k-1}, x_{n_k})}{2}\right]\right\}\\ &\leq \max\left\{\frac{\sigma_p(x_{n_k-1}, x_{n_k})\sigma_p(x_{m_k-1}, x_{m_k})}{\sigma_p(x_{n_k-1}, x_{m_k-1})}, \sigma_p(x_{n_k-1}, x_{n_k}),\right.\\ &\qquad\left.\sigma_p(x_{n_k-1}, x_{m_k-1}), \sigma_p(x_{n_k}, x_{m_k})\right\}.\end{aligned}$$

We should consider the following cases:

(case 1.) If $R(x_{n_k-1}, x_{m_k-1}) = \frac{\sigma_p(x_{n_k-1}, x_{n_k})\sigma_p(x_{m_k-1}, x_{m_k})}{\sigma_p(x_{n_k-1}, x_{m_k-1})}$ (or, $= \sigma_p(x_{n_k-1}, x_{n_k})$). Then, (3.13) becomes

$$\begin{aligned}0 < \psi(r) \longleftarrow \psi(\sigma_p(x_{n_k}, x_{m_k})) &\leq \alpha(x_{n_k-1}, x_{m_k-1})\psi(\sigma_p(x_{n_k}, x_{m_k}))\\ &< h(x_{n_k-1}, x_{m_k-1})\psi(R(x_{n_k-1}, x_{m_k-1}))\\ &\leq \psi(R(x_{n_k-1}, x_{m_k-1})) \longrightarrow \psi(0) \text{ as } k \to \infty\ \Rightarrow\ \text{contradiction.}\end{aligned}$$

(case 2.) If $R(x_{n_k-1}, x_{m_k-1}) = \sigma_p(x_{n_k}, x_{m_k})$. Then, (3.13) becomes

$$\begin{aligned}&\psi(\sigma_p(x_{n_k}, x_{m_k})) \leq \alpha(x_{n_k-1}, x_{m_k-1})\psi(\sigma_p(x_{n_k}, x_{m_k}))\\ &< h(x_{n_k-1}, x_{m_k-1})\psi(R(x_{n_k-1}, x_{m_k-1})) \leq \psi(\sigma_p(x_{n_k}, x_{m_k}))\ \Rightarrow\ \text{contradiction.}\end{aligned}$$

Therefore, $R(x_{n_k-1}, x_{m_k-1})$ must equal $\sigma_p(x_{n_k-1}, x_{m_k-1})$ and then

$$\sigma_p(x_{n_k}, x_{m_k}) \leq \sigma_p(x_{n_k-1}, x_{m_k-1}) \leq R_k(\text{or, } R_k - 1).$$

Thus,

$$\lim_{k\to\infty} \sigma_b(x_{n_k-1}, x_{m_k-1}) = r. \tag{3.14}$$

On account of the above observations, we use (3.13) and then (ζ_3) to obtain

$$\begin{aligned}\psi(r) \leftarrow \psi(\sigma_p(x_{n_k}, x_{m_k})) &\leq \alpha(x_{n_k-1}, x_{m_k-1})\psi(\Omega(\sigma_p(x_{n_k}, x_{m_k})))\\ &< h(x_{n_k-1}, x_{m_k-1})\psi(\sigma_p(x_{n_k-1}, x_{m_k-1})) \leq \psi(\sigma_p(x_{n_k-1}, x_{m_k-1})) \longrightarrow \psi(r) \text{ as } k\to\infty.\end{aligned}$$

Hence,

$$\begin{aligned}0 &\leq \limsup_{n\to\infty} \zeta\big(\alpha(x_{n_k-1}, x_{m_k-1})\psi(\sigma_p(x_{n_k}, x_{m_k})), h(x_{n_k-1}, x_{m_k-1})\psi(\sigma_p(x_{n_k-1}, x_{m_k-1}))\big)\\ &< 0,\end{aligned}$$

which gives a contradiction, then $r = 0$. Thus, (3.9) holds true and the sequence $\{x_n\}$ is σ_p-Cauchy. By the completeness of (X, σ_p), there exists $x \in X$ such that

$$\lim_{n\to\infty} \sigma_p(x_n, x) = \sigma_b(x, x) = \lim_{n,m\to\infty} \sigma_b(x_n, x_m) = 0. \tag{3.15}$$

Now, consider the subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that

$$\sigma_p(x_{n_k}, x) > 0 \text{ and } \sigma_p(x_{n_k+1}, Tx) > 0, \quad \forall\, n_k \geq k \in \mathbb{N}.$$

Lemma 2.1, together with (3.15) imply that

$$\lim_{k\to\infty} \sigma_p(x_{n_k}, x) = 0. \tag{3.16}$$

To prove that $x = Tx$, we have to distinguish two cases:

(case 1.) If $\sigma_p(x_{n_k+1}, Tx) \leq \sigma_p(x_{n_k}, x)$.
In this case, we reach our desired result as follows:

$$\sigma_p(x, Tx) \leq \Omega\big[\sigma_p(x, x_{n_k+1}) + \sigma_p(x_{n_k+1}, Tx)\big] \to 0 \text{ as } n \to \infty. \tag{3.17}$$

Hence, $\sigma_p(x, Tx) = 0 \Rightarrow x = Tx$.

(case 2.) If $\sigma_p(x_{n_k+1}, Tx) > \sigma_p(x_{n_k}, x)$. In this case, we have

$$\begin{aligned}\frac{\Omega^{-1}}{2}\left[\frac{\sigma_p(x_{n_k}, Tx) + \sigma_p(x, x_{n_k+1})}{2}\right] &\leq \frac{\Omega^{-1}}{2}\left[\Omega\big(\sigma_p(x_{n_k+1}, Tx) + \sigma_p(x, Tx)\big)\right]\\ &\leq \max\{\sigma_p(x_{n_k+1}, Tx), \sigma_p(x, Tx)\}.\end{aligned}$$

Since X is $\alpha-$regular and $x_{n_k} \to x$ in X, then $\alpha(x_{n_k}, x) \geq 1$ and then apply (3.1) to obtain

$$\begin{aligned}0 &\leq \zeta\big(\alpha(x_{n_k}, x)\psi(\Omega(\sigma_p(x_{n_k+1}, Tx))), h(x_{n_k}, x)\psi(R(x_{n_k}, x))\big)\\ &< h(x_{n_k}, x)\psi(R(x_{n_k}, x)) - \alpha(x_{n_k}, x)\psi(\Omega(\sigma_p(x_{n_k+1}, Tx))),\end{aligned} \tag{3.18}$$

where

$$R(x_{n_k}, x) \leq \max\left\{\frac{\sigma_p(x_{n_k}, x_{n_k+1})\sigma_p(x, Tx)}{1 + \sigma_p(x_{n_k}, x)}, \sigma_p(x_{n_k}, x_{n_k+1}), \sigma_p(x, Tx), \sigma_p(x_{n_k+1}, Tx)\right\}.$$

We have three subcases:

1. If $R(x_{n_k},x)=\frac{\sigma_p(x_{n_k},x_{n_k+1})\sigma_p(x,Tx)}{1+\sigma_p(x_{n_k},x)}$ $\big($or, $\sigma_p(x_{n_k},x_{n_k+1})\big)$, then equation (3.18) implies

$$\begin{aligned}&\psi(\sigma_p(x_{n_k+1},Tx))\leq\alpha(x_{n_k},x)\psi(\Omega(\sigma_p(x_{n_k+1},Tx)))\\&<h(x_{n_k},x)\psi(R(x_{n_k},x))\leq\psi(R(x_{n_k},x))\to\psi(0)=0 \text{ as } k\to\infty,\\&\Rightarrow\ \lim_{k\to\infty}\sigma_p(x_{n_k+1},Tx)=0,\\&\Rightarrow\ \sigma_p(x,Tx)\leq\Omega\big[\sigma_p(x,x_{n_k+1})+\sigma_p(x_{n_k+1},Tx)\big]\to 0\\&\Rightarrow\ x=Tx.\end{aligned}$$

2. If $R(x_{n_k},x)\leq\sigma_p(x,Tx)$. Then, Lemma 2.2 and Eq. (3.18) yield that

$$\begin{aligned}\psi(\sigma_p(x,Tx))&\leq\psi(\Omega(\sigma_p(x_{n_k+1},Tx)))\leq\alpha(x_{n_k},x)\psi(\Omega(\sigma_p(x_{n_k+1},Tx)))\\&<h(x_{n_k},x)\psi(R(x_{n_k},x))\leq\psi(\sigma_p(x,Tx))\ \Rightarrow\ \text{contradiction.}\end{aligned}$$

3. If $R(x_{n_k},x)\leq\sigma_p(x_{n_k+1},Tx)$. Then, Eq. (3.18) implies

$$\begin{aligned}\psi(\sigma_p(x_{n_k+1},Tx))&\leq\alpha(x_{n_k},x)\psi(\Omega(\sigma_p(x_{n_k+1},Tx)))\\&<h(x_{n_k},x)\psi(\sigma_p(x_{n_k+1},Tx))\leq\psi(\sigma_p(x_{n_k+1},Tx))\ \Rightarrow\ \text{contradiction.}\end{aligned}$$

Therefor, in all cases we get that x is a fixed point of T. To show that this fixed point in unique, suppose that $y\in X$ is another fixed point of T and apply (3.1) to get the opposite.

$$\begin{aligned}0&\leq\zeta\big(\alpha(x,y)\psi(\Omega(\sigma_p(Tx,Ty))),h(x,y)\psi(R(x,y))\big)\\&<h(x,y)\psi(R(x,y))-\alpha(x,y)\psi(\Omega(\sigma_p(x,y))).\end{aligned}$$

Further arguments such as that above we can arrive at $x=y$. Hence, we end up with the uniqueness of the fixed point of T. □

Example 3.1. *Let $X=[0,1]$ and $\sigma_p:X\to X$ be given as*

$$\sigma_p(x,y)=e^{(x+y)^3}-1,:\ \forall\, x,y\in X.$$

Then, (X,σ_p) is a $p-$metric-like space with $\Omega(t)=e^{4t}-1$ (see, Example 2.2). Define $\alpha:X\times X\to\mathbb{R}^+$, $h:X\times X\to(0,1]$, $T:X\to X$, $\psi\in\Psi$ and $\zeta\in\mathcal{Z}$ by

$$\begin{aligned}&\alpha(x,y)=e^{x-y}, \qquad h(x,y)=1, \qquad Tx=\frac{x}{10}\ \forall\, x,y\in X,\\&\psi(t)=\ln\big(1+\frac{1}{4}\ln(1+t)\big), \qquad \zeta(t,s)=\frac{s}{2}-t.\end{aligned}$$

Since the only Cauchy sequence in X is $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ which satisfies $\alpha(x_n,x_{n+1})\geq 1,\ \forall\, n$. This sequence converges to $0\in X$ and $\alpha(x_n,0)\geq 1,\ \forall\, n$. Then (X,σ_p) is $\alpha-$complete and $\alpha-$regular $p-$metric-like space. Also, we have

$$\begin{aligned}&\alpha(x,Tx)\geq 1\Rightarrow x\geq Tx\Rightarrow Tx\geq T^2x\Rightarrow\alpha(Tx,T^2x)\geq 1,\\&\alpha(x,Tx)\geq 1 \text{ and } \alpha(x,Tx)\geq 1\Rightarrow x\geq y \text{ and } y\geq Ty\Rightarrow x\geq Ty\Rightarrow\alpha(x,Ty)\geq 1.\end{aligned}$$

Therefore, T is a triangular $\alpha-$orbital admissible mapping. Now, we show that T also satisfy the contractive condition (3.1).

$$\begin{aligned}
\alpha(x,y)\psi\big(\Omega(\sigma_p(Tx,Ty))\big) &\leq e^{x-y}\psi\big(e^{4[e^{(\frac{x}{10}+\frac{y}{10})^3}-1]}-1\big) \\
&\leq e^{x-y}\ln\big(1+\frac{1}{4}\ln(e^{4[e^{(\frac{x}{10}+\frac{y}{10})^3}-1]})\big) \\
&\leq e^{x-y}(\frac{x}{10}+\frac{y}{10})^3 \\
&\leq \frac{e^{x-y}}{10^3}(x+y)^3 \\
&\leq \frac{e^{x-y}}{10^3}100\ln\big(1+\frac{1}{4}(x+y)^3\big) \\
&\leq \frac{e^{x-y}}{10^3}100\ln\big(1+\frac{1}{4}\ln[1+e^{(x+y)^3}-1]\big) \\
&\leq \frac{e^{x-y}}{10^3}100\ln\big(1+\frac{1}{4}\ln[1+\sigma_p(x,y)]\big) \\
&\leq \frac{e^{x-y}}{10^3}100\ln\big(1+\frac{1}{4}\ln[1+R(x,y)]\big) \\
&\leq \frac{h(x,y)}{2}\psi\big(R(x,y)\big), \\
\Rightarrow\ 0 &\leq \zeta\Big(\alpha(x,y)\psi\big(\Omega(\sigma_p(Tx,Ty)),h(x,y)\psi\big(R(x,y)\big)\Big).
\end{aligned}$$

Hence, all conditions of Theorem 3.1 are satisfied and T has one fixed point $0 \in X$.

Example 3.2. *Let $X = [1,3]$ and $\sigma_p : X \to X$ be given as*

$$\sigma_p(x,y) = e^{(x+y)^2}\sec^{-1} e^{(x+y)^2},:\ \forall\, x,y \in X.$$

Then, (X,σ_p) is a $p-$metric-like space with $\Omega(t) = e^{2t}\sec^{-1} e^{2t}$ (see, Example 2.2). Define $\alpha : X \times X \to \mathbb{R}^+$, $h : X \times X \to (0,1]$, $T : X \to X$, $\psi \in \Psi$ and $\zeta \in \mathcal{Z}$ by

$$\begin{aligned}
&\alpha(x,y) = e^{x-y}, \qquad h(x,y) = 1, \qquad Tx = \frac{x}{10}\ \forall\, x,y \in X, \\
&\psi(t) = \ln\big(\frac{1}{\pi}\ln(\frac{2}{\pi}t)\big), \qquad \zeta(t,s) = \frac{s}{2} - t.
\end{aligned}$$

Indeed, (X,σ_p) is $\alpha-$complete and $\alpha-$regular $p-$metric-like space and T is a triangular $\alpha-$orbital admissible mapping. By using the fact that $\sec^{-1} x \leq \frac{\pi}{2}$, $\forall x \geq 1$, we show that

T also satisfy the contractive condition (3.1).

$$\begin{aligned}
\alpha(x,y)\psi\big(\Omega(\sigma_p(Tx,Ty))\big) &\le e^{x-y}\psi\big(\Omega[e^{(\frac{x}{10}+\frac{y}{10})^2}\sec^{-1}e^{(\frac{x}{10}+\frac{y}{10})^2}]\big)\\
&\le e^{x-y}\psi\big(e^{2[e^{(\frac{x}{10}+\frac{y}{10})^2}\sec^{-1}e^{(\frac{x}{10}+\frac{y}{10})^2}]}\sec^{-1}e^{2[e^{(\frac{x}{10}+\frac{y}{10})^2}\sec^{-1}e^{(\frac{x}{10}+\frac{y}{10})^2}]}\big)\\
&\le e^{x-y}\ln\big(\frac{1}{\pi}\ln(\frac{2}{\pi}e^{2[e^{(\frac{x}{10}+\frac{y}{10})^2}\sec^{-1}e^{(\frac{x}{10}+\frac{y}{10})^2}]}\frac{\pi}{2})\big)\\
&\le e^{x-y}\ln\big(\frac{1}{\pi}\big(2[e^{(\frac{x}{10}+\frac{y}{10})^2}\sec^{-1}e^{(\frac{x}{10}+\frac{y}{10})^2}]\big)\big)\\
&\le e^{x-y}\ln\big(\frac{1}{\pi}\big(2[e^{(\frac{x}{10}+\frac{y}{10})^2}\frac{\pi}{2}]\big)\big)\\
&\le \frac{e^{x-y}}{10^2}(x+y)^2\\
&\le \frac{e^{x-y}}{10^2}18\ln\big(\frac{1}{\pi}\ln(\frac{2}{\pi}e^{(x+y)^2}\sec^{-1}e^{(x+y)^2}\big)\\
&\le \frac{e^{x-y}}{10^2}18\ln\big(\frac{1}{\pi}\ln(\frac{2}{\pi}\sigma_p(x,y))\big)\\
&\le \frac{h(x,y)}{2}\psi\big(R(x,y)\big),\\
\Rightarrow\ 0 &\le \zeta\Big(\alpha(x,y)\psi\big(\Omega(\sigma_p(Tx,Ty))\big), h(x,y)\psi\big(R(x,y)\big)\Big).
\end{aligned}$$

Hence, all conditions of Theorem 3.1 are satisfied and T has one fixed point $0 \in X$.

Remark 3.1. *In Theorem 3.1, we did not use the property (2.3) of the mapping h defined in* [18]*. Moreover, we can use this property instead of ζ_2 to obtain the same result. Also, we can replace the $\alpha-$regular property of X by the continuity of T.*

Taking $\zeta(t,s) = \lambda s - t, \quad \lambda \in [0,1)$, we obtain the following corollary which extends Theorem 2.2 in [18].

Corollary 3.1. *Let (X,σ_p) be a $p-$metric-like space and $T : X \to X$ be a triangular $\alpha-$orbital admissible mapping. Suppose that for all $x,y \in X$ with $\alpha(x,y) \ge 1$,*

$$\alpha(x,y)\psi(\Omega(\sigma_p(Tx,Ty))) \le \lambda h(x,y)\psi(\sigma_p(x,y)), \tag{3.19}$$

where $\psi \in \Psi$, $h : X \times X \to [0,1)$ and

$$R(x,y) = \max\left\{\frac{\sigma_p(x,Tx)\sigma_p(y,Ty)}{\sigma_p(x,y)}, \sigma_p(x,y), \sigma_p(x,Tx), \sigma_p(y,Ty), \frac{\Omega^{-1}}{2}\left[\frac{\sigma_p(x,Ty)+\sigma_p(y,Tx)}{2}\right]\right\}.$$

Consider that the following properties hold true:

(a) *(X,σ_p) is $\alpha-$complete and $\alpha-$regular.*
(b) *For all $x,y \in Fix(T)$, we have $\alpha(x,y) \ge 1$, where $Fix(T)$ denotes the set of fixed points of T.*

Moreover, if there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$, then T has a unique fixed point.

In fact, considering new examples of $p-$metric-like such as those ones given in Example 2.2 yields some new results which generalize an extend many recent results in $b-$metric-like spaces and other metric type spaces.

Corollary 3.2. *Let (X,σ_b) be a $b-$metric-like space with parameter $s\geq 1$ and $T:X\to X$ be a triangular $\alpha-$orbital admissible mapping. Suppose that for all $x,y\in X$ with $\alpha(x,y)\geq 1$,*

$$\zeta\big(\alpha(x,y)\psi(\sinh(s\sinh(\sigma_b(Tx,Ty)))),h(x,y)\psi(\sinh(\sigma_b(x,y)))\big)\geq 0, \tag{3.20}$$

where $\zeta\in\mathcal{Z}$, $\psi\in\Psi$, $h:X\times X\to[0,1)$ and

$$R(x,y)=\max\left\{\frac{\sinh(\sigma_b(x,Tx))\sinh(\sigma_b(y,Ty))}{\sinh(\sigma_b(x,y))},\sinh(\sigma_b(x,y)),\sinh(\sigma_b(x,Tx)),\right.$$
$$\left.\sinh(\sigma_b(y,Ty)),\frac{\sinh^{-1}}{2s}\left[\frac{\sinh(\sigma_b(x,Ty))+\sinh(\sigma_b(y,Tx))}{2}\right]\right\}.$$

Consider that the following properties hold true:

(a) *(X,σ_p) is $\alpha-$complete and $\alpha-$regular.*
(b) *For all $x,y\in Fix(T)$, we have $\alpha(x,y)\geq 1$, where $Fix(T)$ denotes the set of fixed points of T.*

Moreover, if there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$, then T has a unique fixed point.

Let $(X,\preceq,\sigma_b)$ be a partially ordered $p-$metric-like space and $\alpha:X\times X\to\mathbb{R}^+$ be given as:

$$\alpha(x,y)=\begin{cases}1, & x\preceq y;\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, we can derive the following important result in the framework of partially ordered $p-$metric-like spaces.

Corollary 3.3. *Let (X,σ_p) be a complete $p-$metric-like space and $T:X\to X$ be a nondecreasing mapping such that for all $x,y\in X$ with $x\preceq y$, it follows that*

$$\zeta\big(\alpha(x,y)\psi(\Omega(\sigma_p(Tx,Ty))),h(x,y)\psi(\sigma_p(x,y))\big)\geq 0, \tag{3.21}$$

where $\zeta\in\mathcal{Z}$, $\psi\in\Psi$, $h:X\times X\to[0,1)$ and

$$R(x,y)=\max\left\{\frac{\sigma_p(x,Tx)\sigma_p(y,Ty)}{\sigma_p(x,y)},\sigma_p(x,y),\sigma_p(x,Tx),\right.$$
$$\left.\sigma_p(y,Ty),\frac{\Omega^{-1}}{2}\left[\frac{\sigma_p(x,Ty)+\sigma_p(y,Tx)}{2}\right]\right\}.$$

Consider that the following properties hold true:

(a) *X satisfy the following property*

$$\textit{If a nondecreasing sequence } \{x_n\}\to x\in X, \textit{ then } x_n\preceq x,\ \forall\, n. \tag{3.22}$$

(b) *For every two fixed points x and y of T, $x\preceq y$.*

Moreover, if there exists $x_0\in X$ such that $x_0\preceq Tx_0$, then T has a unique fixed point.

Taking $\psi(t)=t$, $\zeta(t,s)=\lambda s-t$, $\forall\, t,s\in\mathbb{R}^+,\lambda\in[0,1)$, $\alpha(x,y)=h(x,y)=1$, $\forall\, x,y\in X$, we obtain an extension of Banach contraction principle in a $p-$metric-like space.

Corollary 3.4. *Let (X, σ_p) be a complete $p-$metric-like space and $T : X \to X$ be such that*

$$\sigma_p(Tx, Ty) \leq \lambda \sigma_p(x, y), \ \ \forall\, x, y \in X. \tag{3.23}$$

Then T has a unique fixed point.

4. **Fractional Differential Equations**

By virtue of the results obtained in the previous section, we give an existence theorem for a solution of problem (1.1) that belongs to $X = C(J, \mathbb{R})$. A thing that we must admit is that the considered problem is inspired in [18] but under different conditions.

Let $X = C(J, \mathbb{R})$ be the set of continuous real functions defined on $J = [0, \tau],\ \tau > 0$ and $\sigma_p : X \times X \to \mathbb{R}^+$ be given as

$$\sigma_p(x, y) = \sigma(x, y) \cosh \sigma(x, y), \ \ \forall\, x, y \in X,$$

where

$$\sigma(x, y) = \max_{t \in J}\big(\left|x(t)\right| + |y(t)|\big), \ \ \forall\, x, y \in X,\ t \in J.$$

Further, we endow (X, σ_p) with an order

$$x \succeq y \ \ \Leftrightarrow \ \ x(t) \geq y(t),\ \forall\, t \in J$$

and then define $\alpha : X \times X \to \mathbb{R}$ as

$$\alpha(x, y) = \begin{cases} \frac{1}{K_\rho}, & x \succeq y; \\ 0, & \text{otherwise}, \end{cases}$$

where

$$K_\rho = \frac{n\tau^{n-1}}{\rho^\beta \left|n\tau^{n-1}e^{-\lambda\tau} - r^n\right|}\big[E_{1,\beta+2}(\lambda, r) + E_{1,\beta+1}(\lambda, \tau)\big] + \frac{1}{\rho^\beta} E_{1,\beta+1}(\lambda, \tau).$$

Here, $E_{\alpha,\beta}(\lambda, z)$ is the modified version of the Mittag-Leffler function defined by

$$E_{\alpha,\beta}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\alpha k+\beta-1}}{\Gamma(\alpha k + \beta)}, \ \ (0 \neq \lambda \in \mathbb{R},\ z, \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0).$$

Obviously, (X, σ_p) is $\alpha-$regular and $\alpha-$complete $p-$metric-like space with

$$\Omega(t) = t \cosh t.$$

In what follows, we provide an integral representation for the solution of our considered problem that will be needed in the sequel.

Lemma 4.1. *The function $x \in X$ is a solution of problem (1.1) if and only if,*

$$\begin{aligned} x(t) = {} & \frac{nt^{n-1}e^{-\lambda t}}{\rho^\beta \Gamma(\beta)\big[n\tau^{n-1}e^{-\lambda\tau} - r^n\big]} \\ & \left[\int_0^r \int_0^s e^{\lambda z}(s - z)^{\beta-1} f(z, x(z))dzds - \int_0^\tau e^{-\lambda(\tau - s)}(\tau - s)^{\beta-1} f(s, x(s))ds\right] \\ & + \frac{1}{\rho^\beta \Gamma(\beta)} \int_0^t e^{-\lambda(t-s)}(t - s)^{\beta-1} f(s, x(s))ds. \end{aligned} \tag{4.1}$$

Proof. We may reduce (1.1) to the equivalent integral form given in (4.1) by applying the operator ${}_0I^{\beta,\rho}$ on both sides of equation (1.1) and making use of Lemma 2.3.

$$x(t)=\sum_{k=0}^{n-1}\frac{(D^{k,\rho}x)(0)}{\rho^k\Gamma(k+1)}t^k e^{\frac{\rho-1}{\rho}t}+{}_0I^{\beta,\rho}f\big(t,x(t)\big). \tag{4.2}$$

Moreover, the condition $x^{(i)}(t)|_{t=0}=0$ implies $(D^{i,\rho}x)(0)=0,\ \ \forall\, 0\leq i\leq n-2$. Hence,

$$\begin{aligned}x(t)&=\frac{(D^{n-1,\rho}x)(0)}{\rho^{n-1}\Gamma(n)}t^{n-1}e^{\frac{\rho-1}{\rho}t}+{}_0I^{\beta,\rho}f\big(t,x(t)\big)\\&=c_{n-1}t^{n-1}e^{-\lambda t}+{}_0I^{\beta,\rho}f\big(t,x(t)\big).\end{aligned} \tag{4.3}$$

Now, we compute the value of c_{n-1}. From (4.3), we get

$$x(\tau)=c_{n-1}\tau^{n-1}e^{-\lambda\tau}+{}_0I^{\beta,\rho}f\big(\tau,x(\tau)\big) \tag{4.4}$$

$$\begin{aligned}\int_0^r e^{\lambda s}x(s)ds&=c_{n-1}\int_0^r s^{n-1}ds+\int_0^r e^{\lambda s}{}_0I^{\beta,\rho}f\big(s,x(s)\big)ds\\&=c_{n-1}\frac{r^n}{n}+\frac{1}{\rho^\beta\Gamma(\beta)}\int_0^r\int_0^s e^{\lambda z}(s-z)^{\beta-1}f(z,x(z))dzds.\end{aligned} \tag{4.5}$$

By $x(\tau)=\int_0^r e^{\lambda s}x(s)ds$, combining with (4.4) and (4.5), we deduce

$$\begin{aligned}c_{n-1}=&\frac{n}{\rho^\beta\Gamma(\beta)\big[n\tau^{n-1}e^{-\lambda\tau}-r^n\big]}\\&\Big[\int_0^r\int_0^s e^{\lambda z}(s-z)^{\beta-1}f(z,x(z))dzds-\int_0^\tau e^{-\lambda(\tau-s)}(\tau-s)^{\beta-1}f(s,x(s))ds\Big].\end{aligned}$$

Substituting c_{n-1} in equation (4.2), we gain (4.1). Conversely, we apply the operator ${}_0D^{\beta,\rho}$ on both sides of equation (4.2) with Lemma 2.4, Propositions 2.2 and 2.3 to obtain

$$\begin{aligned}{}_0D^{\beta,\rho}x(t)&=\sum_{k=0}^{n-1}\frac{\rho^{\beta-k}(D^{k,\rho}x)(0)}{\Gamma(k+1-\beta)}t^{k-\beta}e^{\frac{\rho-1}{\rho}t}+{}_0D^{\beta,\rho}{}_0I^{\beta,\rho}f\big(t,x(t)\big)\\{}_0^CD^{\beta,\rho}x(t)&=f\big(t,x(t)\big).\end{aligned}$$

Furthermore, we have $X^{(i)}(0)=0,\ \ \forall\, 0\leq i\leq n-2$ and

$$\begin{aligned}\int_0^r e^{\lambda s}x(s)ds=&\Big[\frac{r^n}{\rho^\beta\Gamma(\beta)\big[n\tau^{n-1}e^{-\lambda\tau}-r^n\big]}+\frac{1}{\rho^\beta\Gamma(\beta)}\Big]\int_0^r\int_0^s e^{\lambda z}(s-z)^{\beta-1}f(z,x(z))dzds\\&-\frac{r^n}{\rho^\beta\Gamma(\beta)\big[n\tau^{n-1}e^{-\lambda\tau}-r^n\big]}\int_0^\tau e^{-\lambda(\tau-s)}(\tau-s)^{\beta-1}f(s,x(s))ds\\&=\frac{n\tau^{n-1}e^{-\lambda\tau}}{\rho^\beta\Gamma(\beta)\big[n\tau^{n-1}e^{-\lambda\tau}-r^n\big]}\int_0^r\int_0^s e^{\lambda z}(s-z)^{\beta-1}f(z,x(z))dzds\\&-\frac{r^n}{\rho^\beta\Gamma(\beta)\big[n\tau^{n-1}e^{-\lambda\tau}-r^n\big]}\int_0^\tau e^{-\lambda(\tau-s)}(\tau-s)^{\beta-1}f(s,x(s))ds\\&=x(\tau).\end{aligned}$$

This finishes the proof. $\square$

Define the operator $T : X \to X$ as follows:

$$\begin{aligned} Tx(t) = &\frac{nt^{n-1}e^{-\lambda t}}{\rho^{\beta}\Gamma(\beta)\left[n\tau^{n-1}e^{-\lambda\tau} - r^{n}\right]} \\ &\left[\int_0^r \int_0^s e^{\lambda z}(s-z)^{\beta-1} f(z, x(z))dzds - \int_0^\tau e^{-\lambda(\tau-s)}(\tau-s)^{\beta-1} f(s, x(s))ds\right] \\ &+ \frac{1}{\rho^{\beta}\Gamma(\beta)} \int_0^t e^{-\lambda(t-s)}(t-s)^{\beta-1} f(s, x(s))ds. \end{aligned} \tag{4.6}$$

In view of Lemma 4.1, if $x \in X$ is a fixed point of T then it is a solution of (1.1).

Theorem 4.1. *Assume that:*

(A) *If $x, y \in X$ with $x \succeq y$, then*

$$f(t, x(t)) \geq f(t, y(t))$$

and

$$|f(t, x(t))| + |f(t, y(t))| \leq \frac{1}{4}\sinh^{-1}\left(\sinh^{-1} 2(|x(s)| + |y(s)|)\right), \ \ \forall\, t \in J.$$

(B) *There exists $x_0 \in X$ such that $x_0 \succeq Tx_0$.*
(C) $K_\rho \leq 1$.

Then the problem (1.1) has a solution in $x \in X$.

Proof. Consider the mapping $T : X \to X$ defined by (4.6). It follows from (A) that the mapping T is increasing w. r. t. $\succeq$, hence it is triangular $\alpha-$orbital admissible. Now, from

(B) and (C), we have for all $x, y \in X$ with $x \succeq y$

$$
\begin{aligned}
&\sigma(Tx,Ty) = \max_{t\in J}\big(|Tx(t)| + |Ty(t)|\big)\\
&\leq \max_{t\in J}\Bigg(\frac{nt^{n-1}e^{-\lambda t}}{\rho^{\beta}\Gamma(\beta)\big|n\tau^{n-1}e^{-\lambda\tau} - r^n\big|}\Bigg[\int_0^r\int_0^s e^{\lambda z}(s-z)^{\beta-1}\big[|f(z,x(z))| + |f(z,y(z))|\big]dzds\\
&\qquad + \int_0^\tau e^{-\lambda(\tau-s)}(\tau-s)^{\beta-1}\big[|f(s,x(s))| + |f(s,y(s))|\big]ds\Bigg]\\
&\qquad + \frac{1}{\rho^{\beta}\Gamma(\beta)}\int_0^t e^{-\lambda(t-s)}(t-s)^{\beta-1}\big[|f(s,x(s))| + |f(s,y(s))|\big]ds\Bigg)\\
&\leq \frac{1}{4}\sinh^{-1}\sinh^{-1}(2\sigma(x,y)) \times \max_{t\in J}\Bigg(\frac{nt^{n-1}e^{-\lambda t}}{\rho^{\beta}\Gamma(\beta)\big|n\tau^{n-1}e^{-\lambda\tau} - r^n\big|}\Bigg[\int_0^r\int_0^s e^{\lambda z}(s-z)^{\beta-1}dzds\\
&\qquad + \int_0^\tau e^{-\lambda(\tau-s)}(\tau-s)^{\beta-1}ds\Bigg] + \frac{1}{\rho^{\beta}\Gamma(\beta)}\int_0^t e^{-\lambda(t-s)}(t-s)^{\beta-1}ds\Bigg)\\
&\leq \frac{1}{4}\sinh^{-1}\sinh^{-1}(2\sigma(x,y)) \times \max_{t\in J}\Bigg(\frac{nt^{n-1}e^{-\lambda t}}{\rho^{\beta}\Gamma(\beta)\big|n\tau^{n-1}e^{-\lambda\tau} - r^n\big|}\Bigg[\Gamma(\beta)\sum_{k=0}^{\infty}\lambda^k\frac{r^{k+\beta+1}}{\Gamma(k+\beta+2)}\\
&\qquad + e^{-\lambda\tau}\Gamma(\beta)\sum_{k=0}^{\infty}\lambda^k\frac{\tau^{k+\beta}}{\Gamma(k+\beta+1)}\Bigg] + \frac{1}{\rho^{\beta}\Gamma(\beta)}e^{-\lambda t}\Gamma(\beta)\sum_{k=0}^{\infty}\lambda^k\frac{t^{k+\beta}}{\Gamma(k+\beta+1)}\Bigg)\\
&\leq \frac{1}{4}\sinh^{-1}\sinh^{-1}(2\sigma(x,y))\\
&\Bigg(\frac{n\tau^{n-1}}{\rho^{\beta}\big|n\tau^{n-1}e^{-\lambda\tau} - r^n\big|}\big[E_{1,\beta+2}(\lambda,r) + E_{1,\beta+1}(\lambda,\tau)\big] + \frac{1}{\rho^{\beta}}E_{1,\beta+1}(\lambda,\tau)\Bigg)\\
&\leq K_{\rho}\frac{1}{4}\sinh^{-1}\sinh^{-1}(2\sigma(x,y)).
\end{aligned}
$$

Taking $\psi(r_1) = \sinh^{-1}\sinh^{-1}(2r_1)$, $h(r_1,r_2) = 1$ and $\zeta(r_1,r_2) = \frac{r_2}{2} - r_1$, $\forall\ r_1, r_2 \in \mathbb{R}^+$, we deduce that

$$
\begin{aligned}
&\alpha(x,y)\psi\big(\Omega(\sigma_p(Tx,Ty))\big)\\
&\leq \frac{1}{K_{\rho}}\psi\big(\sigma(Tx,Ty)\cosh(\sigma(Tx,Ty))\cosh(\sigma(Tx,Ty)\cosh(\sigma(Tx,Ty)))\big)\\
&\leq \frac{1}{K_{\rho}}\sinh^{-1}\sinh^{-1}\big(2\sigma(Tx,Ty)\cosh(\sigma(Tx,Ty))\cosh(\sigma(Tx,Ty)\cosh(\sigma(Tx,Ty)))\big)
\end{aligned}
$$

$$\leq \frac{1}{K_\rho} \sinh^{-1}\left(2\sigma(Tx,Ty)\cosh(\sigma(Tx,Ty))\right)$$
$$\leq \frac{1}{K_\rho} 2\sigma(Tx,Ty)$$
$$\leq \frac{1}{2} \sinh^{-1}\sinh^{-1}(2\sigma(x,y))$$
$$\leq h(x,y)\psi(\sigma(x,y)),$$
$$\Rightarrow\ 0 \leq \zeta\bigg(\alpha(x,y)\psi\big(\Omega(\sigma_p(Tx,Ty))\big), h(x,y)\psi(\sigma(x,y))\bigg).$$

All obtained results in this section yield to the fact that all the hypotheses of Theorem 3.1 are satisfied and hence the mapping T has one fixed point in X which is a solution of the integral equation (4.1) that is equivalent to our considered problem (1.1). □

5. Concluding Remarks and Observations

We have applied our fixed point results on the existence and uniqueness of fractional differential equations that include different types of fractional operators named as generalized proportional factional derivatives and with exponential weighted integral boundary conditions.

Due to the singularities found in the traditional fractional operators which are thought to make some difficulties in the modeling process. In future works, we look forward to deal with new types of non-singular fractional operators such as those mentioned in [2, 3, 8, 10, 22, 23]. Some of these operators contain exponential kernels and some of them involve the Mittag-Leffler functions.

References

[1] T. Abdeljawad: On conformable fractional calculus. *J. Comput. Appl. Math.*, **279** (2015), 57–66.
[2] T. Abdeljawad and D. Baleanu: Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. *J. Nonlinear Sci. Appl.*, **10(3)** (2017), 1098-1107.
[3] T. Abdeljawad and D. Baleanu: On fractional derivatives with exponential kernel and their discrete versions. *Rep. Math. Phys.*, **80(1)** (2017), 11-27.
[4] M. A. Alghamdi, N. Hussain and P. Salimi: Fixed point and coupled fixed point theorems on $b-$metric-like spaces. *J. Inequal. Appl.*, 2013, **2013**:402.
[5] J. Alzabut, T. Abdeljawad, F. Jarad and W. Sudsutad: A Gronwall inequality via the generalized proportional fractional derivative with applications. *J. Inequal. Appl.*, (2019) 2019:101.
[6] A. Amini-Harandi: Metric-like spaces, partial metric spaces and fixed points. *Fixed Point Theory Appl.*, 2012, **2012:**204.
[7] D. R. Anderson and D. J. Ulness: Newly defined conformable derivatives. *Adv. Dyn. Sys. App.*, **10(2)** (2015), 109-137.
[8] A. Atangana and D. Baleanu: New fractional derivative with non-local and non-singular kernel. *Thermal Sci.*, **20** (2016), 757-763.
[9] I. A. Bakhtin: The contraction mapping principle in quasi-metric spaces. *Funct. Anal. Unianowsk Gos. Ped. Inst.*,**30** (1989), 26–37.
[10] M. Caputo and M. Fabrizio: A new definition of fractional derivative without singular kernel. *Progr. Fract. Differ. Appl.*, **1** (2015), 73-85.
[11] L. Ćirić, V. Parvaneh and N. Hussain: Fixed point results for weakly $\alpha-$admissible pairs. *Filomat*, **30(14)** (2016), 3697-3713.

[12] S. Czerwik: Contraction mappings in b-metric spaces. *Acta. Math. Inform. Univ. Ostrav.*, **1** (1993), 5–11.

[13] L. Debnath: Recent applications of fractional calculus to science and engineering. . *Int. J. Math. Math. Sci.*, **54** (2003), 3413-3442.

[14] R. Hilfer: *Applications of Fractional Calculus in Physics.* World Scientific Publishing Company, Singapore, New Jersey, Londin and Hong Kong, 2000.

[15] N. Hussain, M. A. Kutbi and P. Salimi: Fixed point theory in $\alpha-$complete metric spaces with applications. *Abstr. Appl. Anal.*, **2014**, Article ID 280817, 11 pages.

[16] N. Hussain, J. R. Roshan, V. Parvaneh and Z. Kadelburg: Fixed points of contractive mappings in $b-$metric-like spaces. *Sci. World J.*, **2014**, Article ID 471827, 15 pages.

[17] F. Jarad, T. Abdeljawad and J. Alzabut: Generalized fractional derivatives generated by a class of local proportional derivatives. *Eur. Phys. J. Special Topics*, **226** (2017), 3457–3471.

[18] E. Karapınar, T. Abdeljawad and F. Jarad: Applying new fixed point theorems on fractional and ordinary differential equations. *Adv. Differ. Equ.*, (2019) 2019:421.

[19] E. Karapınar, P. Kumam and P. Salimi: On $\alpha - \psi-$Meir-Keeler contractive mappings. *Fixed Point Theory Appl.*, 2013, **2013**:94.

[20] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh: A new definition of fractional derivative. *J. Comput. Appl. Math.*, **264** (2014), 65-70.

[21] M. A. Khan, A. Atangana: Modeling the dynamics of novel coronavirus (2019-nCov) with fractional derivative. *Alex. Eng. J.*, (2020, in press). https://doi.org/10.1016/j.aej.2020.02.033

[22] A. Khan, H. Khan, J. F. Gómez-Aguilar and T. Abdeljawad: Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel. *Chaos Soliton Fract.*, **127** (2019), 422-427.

[23] H. Khan, Y. Li, A. Khan and Az. Khan: Existence of solution for a fractional-order Lotka-Volterra reaction-diffusion model with Mittag-Leffler kernel. *Math. Meth. Appl. Sci.*, **42(9)** (2019), 3377-3387.

[24] M. S. Khan, M. Swaleh and S. Sessa: Fixed point theorems by altering distancces between the points. *Bull. Aust. Math. Soc.*, **30** (1984), 1–9.

[25] F. Khojasteh, S. Shukla and S. Radenović: A new approach to the study of fixed point theory for simulation functions. *Filomat*, **29(6)** (2015), 1189-1194.

[26] R. L. Magin: *Fractional Calculus in Bioengineering.* Begell House, Redding, 2006.

[27] F. Mainardi : *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models.* World Scientific, Singapore, 2010.

[28] S. G. Matthews: Partial metric topology. *Ann. New York Acad. Sci.*, **728** (1994), 183-197.

[29] K. M. Owolabi, A. Atangana and A. Akgul: Modelling and analysis of fractal-fractional partial differential equations: Application to reaction-diffusion model. *Alex. Eng. J.*, (2020, in press). https://doi.org/10.1016/j.aej.2020.03.022

[30] V. Parvaneh, A. Dinmohammadi and Z. Kadelburg: Coincidence point results for weakly $\alpha-$admissible pairs in extended $b-$metric spaces. *J. Math. Anal.*, **8(6)** (2017), 74–89.

[31] V. Parvaneh, S. J. H. Ghoncheh: Fixed Points of $(\psi, \varphi)_\Omega-$Contractive Mappings in Ordered $P-$Metric Spaces. *Global Anal. Discrete Math.*, **4(1)** (2020), 15–29.

[32] V. Parvaneh and Z. Kadelburg: Extended partial $b-$metric spaces and some fixed point results. *Filomat*, **32(8)** (2018), 2837-2850.

[33] V. Parvaneh and Z. Kadelburg: Fixed points of $JSHR-$contractive type mappings in extended $b-$metric-like spaces. *Vietnam J. Math.*, **47** (2019), 387–401.

[34] O. Popescu: Some new fixed point theorems for $\alpha-$Geraghty contraction type maps in metric spaces. *Fixed Point Theory Appl.*, 2014, 2014:190.

[35] B. Samet, C. Vetro and P. Vetro: Fixed point theorems for $\alpha-\psi-$contractive type mappings. *Nonlinear Anal.*, **75(4)** (2012), 2154–2165.

[36] S. Shukla: Partial $b-$metric spaces and fixed point theorems. *Mediterr. J. Math.*, **11** (2014), 703-711.