A Nonhomogeneous Boundary-Valued Problem for the coupled KDV system

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Abstract. In this paper, we study the initial-boundary-value problem (IBVP) for coupled Korteweg-de Vries equations posed on a finite interval with nonhomogeneous boundary conditions. We overcome the requirement for stronger smooth boundary conditions in the traditional method via the Laplace transform. Our approach uses the strong Kato smoothing property and the contraction mapping principle.

Keywords: the coupled Korteweg-de Vries equations; globally well-posed.

1 Introduction and main results

In this paper, we investigate the initial-boundary value problem (IBVP) of the coupled Korteweg-de Vries (cKdv) system:

$$u_t + u_{xxx} + 6uu_x - 2bvv_x = 0 \tag{1.1}$$

$$v_t + v_{xxx} + 3uv_x = 0 \tag{1.2}$$

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posed on $\{(x,t) \in \Omega \times R^+\}$, where $\Omega = (m,n)$ is an interval in R, and b is a real positive constant. Moreover, the system (1.1)-(1.2) has a certain application for describing the interaction of two long waves with different dispersion relation [1] [2]. The initial and boundary conditions of (1.1)-(1.2) are given by (without loss of generality, we choose $(m,n) = (0,1)$):

$$u(x,0) = u_0(x), v(x,0) = v_0(x) \tag{1.3}$$

$$u(0,t) = h_{11}(t), u(1,t) = h_{12}(t), u_x(1,t) = h_{13}(t) \tag{1.4}$$

$$v(0,t) = h_{21}(t), v(1,t) = h_{22}(t), v_x(1,t) = h_{23}(t) \tag{1.5}$$

where $u_0(x), v_0(x), h_{ij}(t)(i = 1,2; j = 1,2,3)$ are given functions.

For the KdV equation, Bubnov (1979, 1980) studied the general two-point boundary-value problem posed on a finite interval. Bona and Dougalis (1980) gave the related work on the BBM-equation. Zhang (1994) studied the Dirichlet boundary conditions posed on a finite interval and show the global well-posedness in the space $H^{3k+1}(0,1)$ for $k = 0, 1, \ldots.$ of the KdV system. Colin and Ghidaglia (2001) considered the following initial-boundary value problem:

$$u_t + uu_x + u_{xxx} = 0, u(x,0) = \phi(x), x \in (0,1), t \geq 0$$

$$u(0,t) = h_1(t), u_x(1,t) = h_2(t), u_{xx}(1,t) = h_2(t), t \geq 0$$

and gave the locally well-posed on the space $H^1(0,1)$.

In this article, we will prove the well-posedness of the nonhomogeneous boundary-value problem (1.1)-(1.5) in $H^s(0,r)\times H^s(0,r)$ with the assumption that the initial condition lies in the product space $H^s(0,r)\times H^s(0,r)$ and the boundary condition is drawn from the product space $H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$. Without loss of generality, we choose underlying spatial domain $(0,r) = (0,1)$. Moreover, if (u,v) is a $C^\infty-$ smooth solution of the IBVP (1.1)-(1.5), its initial conditions $u_0(x), v_0(x)$ and its boundary conditions $h_{ij}(t), (i = 1,2; j = 1,2,3)$ are assumed to satisfy the s-compatibility conditions:

$$u_{0k}(0) = h_{11}^{(k)}(0), u_{0k}(1) = h_{12}^{(k)}(0), u'_{0k}(1) = h_{13}^{(k)}(0)$$

$$v_{0k}(0) = h_{21}^{(k)}(0), v_{0k}(1) = h_{22}^{(k)}(0), v'_{0k}(1) = h_{23}^{(k)}(0)$$

where $k = 0, 1, \ldots,$ and $h_{ij}^{(k)}(t)$ is the k-th order derivative of h_j and

$$\begin{aligned}
&u_{00}(x) = u_0(x), v_{00}(x) = v_0(x) \\
&u_{0k}(x) = -(u'''_{0(k-1)}(x) + u'_{0(k-1)}(x) + \sum_{j=0}^{k-1}(u_{0j}(x)u_{0(k-1-j)}(x))') \\
&v_{0k}(x) = -(v'''_{0(k-1)}(x) + v'_{0(k-1)}(x) + \sum_{j=0}^{k-1}(v_{0j}(x)v_{0(k-1-j)}(x))')
\end{aligned}$$

where $k = 0, 1, \ldots$ Then we give the following definition:

Definition 1.1 *(s-compatibility) Let $T > 0$ and $s > 0$ be given. $(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3) = (u_0, v_0, h_{11}, h_{21}, h_{12}, h_{22}, h_{13}, h_{23}) \in X_{s,T} \times X_{s,T}$ is said to be s-compatible if*

$$u_{0k}(0) = h_{11}^{(k)}(0), u_{0k}(1) = h_{12}^{(k)}(0), v_{0k}(0) = h_{21}^{(k)}(0), v_{0k}(1) = h_{22}^{(k)}(0) \tag{1.6}$$

holds for $k = 0, 1, \ldots, [\frac{s}{3} - 1]$ when $s - 3[\frac{s}{3}] \leq \frac{1}{2}$, or (1.6) holds for $k = 0, 1, \ldots, [\frac{s}{3}]$ when $\frac{1}{2} < s - 3[\frac{s}{3}] \leq \frac{3}{2}$, and

$$u_{0k}(0) = h_{11}^{(k)}(0), u_{0k}(1) = h_{12}^{(k)}(0), u'_{0k}(1) = h_{13}^{(k)}(0)$$

$$v_{0k}(0) = h_{21}^{(k)}(0), v_{0k}(1) = h_{22}^{(k)}(0), v'_{0k}(1) = h_{23}^{(k)}(0)$$

holds for $k = 0, 1, \ldots, [\frac{s}{3}]$ when $s - 3[\frac{s}{3}] > \frac{3}{2}$, we adopt the convention that Eq.(1.6) is vacuous if $[\frac{s}{3}] - 1 < 0$.

Then we give the main results of this article as the following two theorems:

Theorem 1.2 *(Local well-posedness) Assume that $T > 0$, $s \geq 0$, $(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3) = (u_0, v_0, h_{11}, h_{21}, h_{12}, h_{22}, h_{13}, h_{23}) \in X_{s,T} \times X_{s,T}$ is s-compatible. Then there exists a $T^* \in (0, T]$ relying only on the norm of $(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3)$ in the space $X_{s,T} \times X_{s,T}$ such that the system (1.1)-(1.5) admits a unique solution*

$$(u, v) \in (C([0, T^*]; H^s(0, 1)) \cap L_2([0, T^*]; H^{s+1}(0, 1))) \times (C([0, T^*]; H^s(0, 1)) \cap L_2([0, T^*]; H^{s+1}(0, 1)))$$

Theorem 1.3 *(Global well-posedness) Assume that* $T > 0$ *is arbitrary,* $s \geq 0, (\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3) = (u_0, v_0, h_{11}, h_{21}, h_{12}, h_{22}, h_{13}, h_{23}) \in H^s(0,1) \times H^{\mu_1(s)}(0,T) \times H^{\mu_1(s)}(0,T) \times H^{\mu_2(s)}(0,T) \times H^s(0,1) \times H^{\mu_1(s)}(0,T) \times H^{\mu_1(s)}(0,T) \times H^{\mu_2(s)}(0,T)$ *where*

$$\mu_1(s) = \begin{cases} \epsilon + \dfrac{5s+9}{18} & if \quad 0 \leq s \leq 3, \\ \dfrac{s+1}{3} & if \qquad s \geq 3 \end{cases}$$

$$\mu_2(s) = \begin{cases} \epsilon + \dfrac{5s+3}{18} & if \quad 0 \leq s \leq 3, \\ \dfrac{s}{3} & if \qquad s \geq 3 \end{cases}$$

where ϵ *is any positive constant, the IBVP* (1.1)-(1.5) *admits a unique solution*

$$(u, v) \in (C([0,T]; H^s(0,1)) \cap L_2([0,T]; H^{s+1}(0,1))) \times (C([0,T]; H^s(0,1)) \cap L_2([0,T]; H^{s+1}(0,1))).$$

Note that $X_{s,T}$ will be defined later in Section 3, $X_{s,T} = H^s(0,1) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$.

The rest of this article is organized as follows. In Section 2, we give the smooth properties of the associated linear problem. A traditional method to consider the corresponding non-homogeneous boundary-value problem is to render its boundary conditions homogeneous. However, this needs a stronger boundary smoothing property. In order to overcome this requirement, we seek an explicit solution formula via the Laplace transform. In Section 3, we obtain the locally well-posed of (1.1)-(1.5) by using the linear estimates and the fixed point theory. In Section 4, we show the global well-posedness of the system (1.1)-(1.5).

2 Linear estimates and smoothing properties

In this section, we will study the linear system (1.1)-(1.5), and give various smoothing properties. Let $\vec{\omega} = (u, v)^T, \vec{\phi} = (u_0(x), v_0(x))^T, \vec{h_i}(t) = (h_{1i}(t), h_{2i}(t))^T, \vec{F} = (f_1, f_2)^T$, the IBVP (1.1)-(1.5) can be written as

$$\vec{w}_t + \vec{w}_{xxx} = \vec{F} \tag{2.1}$$

$$\vec{w}(x,0) = \vec{\phi}(x) \tag{2.2}$$

$$\vec{w}(0,t) = \vec{h}_1(t), \vec{w}(1,t) = \vec{h}_2(t), \vec{w}_x(1,t) = \vec{h}_3(t) \tag{2.3}$$

then we just need to discuss the linear system (2.1)-(2.3). Firstly, we consider the problem

$$\vec{w}_t + \vec{w}_{xxx} = \vec{0} \tag{2.4}$$

$$\vec{w}(x,0) = \vec{\phi}(x) \tag{2.5}$$

$$\vec{w}(0,t) = \vec{0}, \vec{w}(1,t) = \vec{0}, \vec{w}_x(1,t) = \vec{0} \tag{2.6}$$

with zero forcing and homogeneous boundary conditions. Define A as a linear operator in the space

$$L_2(0,1) \times L_2(0,1)$$

by

$$A\vec{g} = -\vec{g}'''$$

with the domain

$$D(A) = \{\vec{g} \in H^3(0,1) \times H^3(0,1), \vec{g}(0) = \vec{g}(1) = \vec{g}'(1) = \vec{0}\}$$

Then the initial-boundary value problem (2.4)-(2.6) is equivalent to the following abstract evolution equation in $L_2(0,1) \times L_2(0,1)$

$$\frac{d\vec{w}}{dt} = A\vec{w}, \vec{w}(0) = \vec{\phi}$$

By direct computation, we can get that both A and its adjoint A^* are dissipative. Then follows from the standard semi-group theory [4], we know that the operator A is the infinitesimal generator of a C_0-semigroup $P_0(t)$ in the space $L_2(0,1) \times L_2(0,1)$, where $P_0(t) = diag\{p_0(t), p_0(t)\}$. For given $\vec{\phi} \in L_2(0,1) \times L_2(0,1)$, the mild solution $\vec{w} \in C(R^+; L^2(0,1)) \times C(R^+; L^2(0,1))$ to (2.4)-(2.6) can be written in the form of

$$\vec{\omega}(t) = P_0(t)\vec{\phi}$$

For given $\vec{\phi} \in D(A)$, the strong solution $\vec{\omega}(t) = P_0(t)\vec{\phi}$ belong to the smaller space

$$(C(0,\infty; H^3(0,1) \cap C^1(0,\infty; L_2(0,1))) \times (C(0,\infty; H^3(0,1) \cap C^1(0,\infty; L_2(0,1))),$$

then we discuss the global Kato smoothing property as in the following:

Lemma 2.1 *For any $\vec{\phi} \in L_2(0,1) \times L_2(0,1)$, there exist a constant C such that the solution $\vec{w}(t) = P_0(t)\vec{\phi}$ of (2.4)-(2.6) satisfies*

$$\|\vec{w}||_{L_2(0,t;H^1(0,1))\times L_2(0,t;H^1(0,1))} \leq C\|\vec{\phi}\|_{L_2(0,1)\times L_2(0,1)}, t \geq 0 \tag{2.7}$$

Proof. If $\vec{\phi} \in D(A)$, then the strong solution $\vec{w}(t)$ of (2.4)-(2.6) belong to the space $C^1(0,\infty; L_2(0,1)) \times C^1(0,\infty; L_2(0,1))$ and $\vec{w}(t) \in D(A)$for any$t \geq 0$. Multiply (2.4) by $2\vec{w}$, integrate over $(0,t)$ with respect to t and integrate over $(0,1)$ with respect to x. Integrate by part we can get

$$\|\vec{w}(\cdot,t)\|^2_{L_2(0,1)\times L_2(0,1)} + \int_0^t {\vec{w}_x}^2(0,\tau)d\tau = \|\vec{\phi}\|^2_{L_2(0,1)\times L_2(0,1)} \tag{2.8}$$

which gives a boundary smoothing effect. Multiply (2.4) by $2x\vec{w}$, integrate over $(0,t)$ with respect to t and integrate over $(0,1)$ with respect to x. Integrate by part we can get

$$\int_0^1 x\vec{w}^2(x,t)dx + 3\int_0^t\int_0^1 \vec{w}_x^2 dxd\tau = \int_0^1 x\vec{\phi}^2(x)dx \tag{2.9}$$

Combing (2.8) and (2.9), we can get a Kato-type smoothing effect (2.7). By choosing a sequence $\{\vec{\phi}_n\} \subset D(A)$ and using a limiting procedure, we can easily get the general case about $\vec{\phi} \in L_2(0,1) \times L_2(0,1)$. □

Then, we consider the problem

$$\vec{w}_t + \vec{w}_x = \vec{F}(x,t) \tag{2.10}$$

$$\vec{w}(x,0) = \vec{0} \tag{2.11}$$

$$\vec{w}(0,t) = \vec{0}, \vec{w}(1,t) = \vec{0}, \vec{w}_x(1,t) = \vec{0} \tag{2.12}$$

with non-trivial forcing $\vec{F}$ and all three boundary conditions set to zero. Then the initial-boundary value problem (2.10)-(2.12) is equivalent to the following abstract nonhomogeneous evolution equation

$$\frac{d\vec{w}}{dt} = A\vec{w} + \vec{F}, \vec{w}(0) = 0$$

Then follows from the standard semi-group theory [4], for given $\vec{F} \in L_{1,loc}(R^+; L_2(0,1)) \times L_{1,loc}(R^+; L_2(0,1))$, the mild solution $\vec{w} \in C(R^+; L_2(0,1)) \times C(R^+; L_2(0,1))$ to (2.10)-(2.12) can be written in the form of $\vec{w}(t) = \int_0^t P_0(t-\tau)\vec{F}(\tau)d\tau$. For given $\vec{F} \in D(A)$ and $A\vec{F} \in L_{1,loc}(R^+; L_2(0,1)) \times L_{1,loc}(R^+; L_2(0,1)), \vec{w}(t)$ is a strong solution.

Lemma 2.2 *For any $\vec{F} \in L_{1,loc}(R^+; L_2(0,1)) \times L_{1,loc}(R^+; L_2(0,1))$, there exist a constant C such that the solution $\vec{w}(t) = \int_0^t P_0(t-\tau)\vec{F}(\tau)d\tau$ of* (2.10)-(2.12) *satisfies*

$$\|\vec{w}||_{L_2(0,t;H^1(0,1))\times L_2(0,t;H^1(0,1))} \leq C(1+t)^{\frac{1}{2}}\|\vec{F}\|_{L_1(0,t;L_2(0,1))\times L_1(0,t;L_2(0,1))} \tag{2.13}$$

Proof. Without loss of generality, let $\vec{F} \in D(A)$, Multiply (2.10) by $2\vec{w}$, integrate over $(0,t)$ with respect to t and integrate over $(0,1)$ with respect to x, we can obtain

$$\|\vec{w}(\cdot,t)\|^2_{L_2(0,1)\times L_2(0,1)} + \int_0^t \vec{w_x}^2(0,\tau)d\tau \leq C\|\vec{F}\|^2_{L_1(0,t;L_2(0,1))\times L_1(0,t;L_2(0,1))} \tag{2.14}$$

Multiply (2.10) by $2x\vec{w}$, integrate over $(0,t)$ with respect to t and integrate over $(0,1)$ with respect to x, we can obtain

$$\int_0^1 x\vec{w}^2(x,t)dx + 3\int_0^t\int_0^1 \vec{w}_x^2 dxd\tau \leq 2(1+t)\|\vec{F}\|^2_{L_1(0,t;L_2(0,1))\times L_1(0,t;L_2(0,1))} \tag{2.15}$$

Combining (2.14) and (2.15), we can get (2.13). Similar to the proof of Lemma 2.1, we can also get the general case about $\vec{F} \in L_{1,loc}(R^+; L_2(0,1)) \times L_{1,loc}(R^+; L_2(0,1))$. □

Next, we turn to discuss the problem

$$\vec{w}_t + \vec{w}_{xxx} = \vec{0} \tag{2.16}$$

$$\vec{w}(x,0) = \vec{0} \tag{2.17}$$

$$\vec{w}(0,t) = \vec{h}_1(t), \vec{w}(1,t) = \vec{h}_2(t), w_x(1,t) = \vec{h}_3(t) \tag{2.18}$$

with zero forcing but with all three non-trivial boundary condition. By using the Laplace transform, we can get an explicit solution formula of (2.16)-(2.17) [9].

We can convert (2.16) to the following system via the Laplace transform with respect to t.

$$s\hat{\vec{w}}(x,s) + \hat{\vec{w}}_{xxx}(x,s) = \vec{0} \tag{2.19}$$

$$\hat{\vec{w}}(0,s) = \hat{\vec{h}}_1(s), \hat{\vec{w}}(1,s) = h_2\hat{\vec{(}}s), \hat{\vec{w}}_x(1,s) = \hat{\vec{h_3}}(s) \tag{2.20}$$

where

$$\hat{\vec{\omega}}(x,s) = \int_0^{+\infty} e^{-st}\vec{w}(x,t)dt$$

$$\hat{\vec{h}}_j(s) = \int_0^{+\infty} e^{-st}\vec{h}_j(t)dt, j = 1,2,3$$

we can write the solution of (2.19)-(2.20) in the form

$$\hat{\vec{\omega}}(x,s) = \vec{c_1}(s)e^{\lambda_1(s)x} + \vec{c_2}(s)e^{\lambda_2(s)x} + \vec{c_3}(s)e^{\lambda_3(s)x} = \sum_{j=1}^{3} \vec{c_j}(s)e^{\lambda j(s)x}$$

where $\lambda_j = \lambda_j(s), j = 1,2,3$ satisfy the characteristic equation

$$\lambda^3 + s = 0 \tag{2.21}$$

and $\vec{c_j}(s) = (c_{j1}(s), c_{j2}(s))^T$ satisfy the system

$$\begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1(s)} & e^{\lambda_2(s)} & e^{\lambda_3(s)} \\ \lambda_1 e^{\lambda_1(s)} & \lambda_2 e^{\lambda_2(s)} & \lambda_3 e^{\lambda_3(s)} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix} = \begin{pmatrix} \hat{h_{11}} & \hat{h_{12}} \\ \hat{h_{21}} & \hat{h_{22}} \\ \hat{h_{31}} & \hat{h_{32}} \end{pmatrix}$$

by Gramer's rule

$$c_{jp} = \frac{\Delta_{jp}(s)}{\Delta(s)}, j = 1,2,3, p = 1,2$$

where $\Delta(s)$ is the determinant of the coefficient matrix, $\Delta(s) \neq 0$ and Δ_{jp} is the determinant of the matrix with the column j replaced by the vector $(\hat{h}_{1p}(s), \hat{h}_{2p}(s), \hat{h}_{3p}(s))^T, j = 1,2,3; p = 1,2$. By taking the inverse Laplace transform of $\hat{\vec{w}}(x,s)$, we get

$$\begin{aligned} \vec{w}(x,t) &= \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\hat{\vec{w}}(x,s)ds \\ &= (\sum_{j=1}^{3} \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_{j1}(s)}{\Delta(s)}e^{\lambda j(s)}ds, \sum_{j=1}^{3} \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_{j2}(s)}{\Delta(s)}e^{\lambda j(s)}ds)^T \end{aligned}$$

$r > 0$,and the solution of (2.19)-(2.20) also can be written as

$$\vec{w}(x,t) = \vec{w}_1(x,t) + \vec{w}_2(x,t) + \vec{w}_3(x,t) = \sum_{n=1}^{3} \vec{w}_n(x,t)$$

where $\vec{w}_n(x,t)$ is the solution of (2.19)-(2.20) with $\vec{h}_j = \vec{0}, j \neq n, j = 1,2,3$. Thus $\vec{w}_n(x,t)$ can be written as follows:

$$\begin{aligned} &\vec{w}_n(x,t) \\ &= (\sum_{j=1}^{3} \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_{j1,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{n1}(s)ds, \sum_{j=1}^{3} \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_{j2,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{n2}(s)ds)^T \end{aligned}$$

$$\equiv \ [Q_n(t)\overrightarrow{h_n}](x), n=1,2,3 \tag{2.22}$$

where $\Delta_{jp,n}(s)$ are get from $\Delta_{jp}(s)$ by letting $\hat{h}_{np}(t)=1, \hat{h}_{kp}(t)=0, k\neq n, k,n=1,2,3$. We know that

$$\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_{jm,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{nm}(s)ds = lim_{R\to\infty}\int_{r-iR}^{r+iR} e^{st}\frac{\Delta_{jm,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{nm}(s)ds$$

where $j,n = 1,2,3; m = 1,2$. The above integrable functions are analytic on the whole space. According to the Cauchy integral theorem, there are the following characteristics in the connected closed loop Q :

$$\int_Q e^{st}\frac{\Delta_{jm,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{nm}(s)ds = (\int_{l_1}+\int_{l_2}+\int_{l_3}+\int_{l_4})e^{st}\frac{\Delta_{jm,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{nm}(s)ds = 0.$$

Q, l_1, l_2, l_3, l_4 are shown in Fig.1, $a\in[0,r]$.

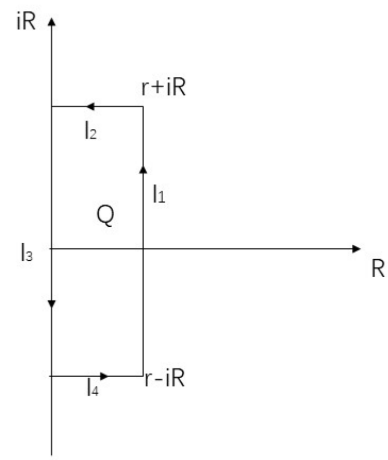

Fig.1

The integrals along the two lines cancel one another. Because we can easily verify

$$e^{(a+iR)t}\frac{\Delta_{jm,n}((a+iR))}{\Delta((a+iR))}e^{\lambda_j((a+iR))x}\hat{h}_{nm}((a+iR)) = e^{(a-iR)t}\frac{\Delta_{jm,n}((a-iR))}{\Delta((a-iR))}e^{\lambda_j((a-iR))x}\hat{h}_{nm}((a-iR)).$$

So

$$\int_{l_2} e^{st}\frac{\Delta_{jm,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{nm}(s)ds + \int_{l_4} e^{st}\frac{\Delta_{jm,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{nm}(s)ds = 0.$$

Then we can obtain

$$\int_{l_1} e^{st}\frac{\Delta_{jm,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{nm}(s)ds+\int_{l_3} e^{st}\frac{\Delta_{jm,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{nm}(s)ds=0.$$

Which means

$$\int_{r-iR}^{r+iR} e^{st}\frac{\Delta_{jm,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{nm}(s)ds=\int_{-iR}^{iR} e^{st}\frac{\Delta_{jm,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{nm}(s)ds=0.$$

So we can also make $s=i\rho^3$ and get the $\vec{w}_n(x,t)$ in the form

$$\begin{aligned}
&\vec{w}_n(x,t)\\
=\;&(\sum_{j=1}^{3}\frac{1}{2\pi i}\int_0^{+i\infty} e^{st}\frac{\Delta_{j1,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{n1}(s)ds+\sum_{j=1}^{3}\frac{1}{2\pi i}\int_{-i\infty}^{0} e^{st}\frac{\Delta_{j1,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{n1}(s)ds,\\
&\sum_{j=1}^{3}\frac{1}{2\pi i}\int_0^{+i\infty} e^{st}\frac{\Delta_{j2,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{n2}(s)ds+\sum_{j=1}^{3}\frac{1}{2\pi i}\int_{-i\infty}^{0} e^{st}\frac{\Delta_{j2,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_{n2}(s)ds)^T\\
\equiv\;&\vec{I}_n(x,t)+\vec{II}_n(x,t), n=1,2,3
\end{aligned}$$

Letting $s=(-i\rho)^3=i\rho^3,\quad \rho\in[1,\infty)$, we can get three roots from the characteristic equation (2.21):

$$\lambda_1^+(\rho)=i\rho, \lambda_2^+(\rho)=-i\rho(\frac{1+i\sqrt{3}}{2}), \lambda_3^+(\rho)=-i\rho(\frac{1-i\sqrt{3}}{2})$$

then we can get $\vec{I}_n(x,t), \vec{II}_n(x,t)$ as follows:

$$\begin{aligned}
&\vec{I}_n(x,t)\\
=\;&(\sum_{j=1}^{3}\frac{1}{2\pi}\int_1^{+\infty} e^{i\rho^3 t}e^{\lambda_j^+(\rho)x}\frac{\Delta_{j1,n}^+(\rho)}{\Delta^+(\rho)}3\rho^2\hat{h}_{n1}^+(\rho)d\rho,\sum_{j=1}^{3}\frac{1}{2\pi}\int_1^{+\infty} e^{i\rho^3 t}e^{\lambda_j^+(\rho)x}\frac{\Delta_{j2,n}^+(\rho)}{\Delta^+(\rho)}3\rho^2\hat{h}_{n2}^+(\rho)d\rho)^T\\
&\vec{II}_n(x,t)\\
=\;&(\sum_{j=1}^{3}\frac{1}{2\pi}\int_1^{+\infty} e^{-i\rho^3 t}e^{\lambda_j^-(\rho)x}\frac{\Delta_{j1,n}^-(\rho)}{\Delta^-(\rho)}3\rho^2\hat{h}_{n1}^-(\rho)d\rho,\sum_{j=1}^{3}\frac{1}{2\pi}\int_1^{+\infty} e^{-i\rho^3 t}e^{\lambda_j^-(\rho)x}\frac{\Delta_{j2,n}^-(\rho)}{\Delta^-(\rho)}3\rho^2\hat{h}_{n2}^-(\rho)d\rho)^T,
\end{aligned}$$

$n=1,2,3$, where $\hat{h}_{n1}^+(\rho)=\hat{h}_{n1}(i\rho^3)$, by replacing s with $i\rho^3$ and $\lambda_j(s)$ with $\lambda_j^+(\rho)$, we give $\Delta^+(\rho)$ and $\Delta_{ji,n}^+(\rho)(i=1,2)$ from $\Delta(s)$and$\Delta_{ji,n}(s), \Delta^-(\rho)=\overline{\Delta^+(\rho)}, \Delta_{ji,m}^-(\rho)=\overline{\Delta_{ji,m}^+}(\rho)$ and $\hat{h}_{ni}^-(\rho)=\overline{\hat{h}_{ni}^+(\rho)}, j=1,2,3; i=1,2$. The next Lemma 2.3 is given by Bona, Sun and Zhang [9].

Lemma 2.3 *For any $f \in L_2(0,\infty)$,let Kf be the function defined by*

$$kf(x) = \int_0^{+\infty} e^{\gamma(\mu)x} f(\mu) d\mu$$

where $\gamma(\mu)$ is a continuous complex-valued function defined on $(0,\infty)$ satisfying the following two conditions:

(i)There exist $\delta > 0, b > 0$ such that

$$\sup_{0<\mu<\delta} \frac{|Re\gamma(\mu)|}{\mu} \geq b.$$

(ii)There exist a complex number $\alpha + i\beta$ such that

$$\lim_{\mu\to\infty} \frac{\gamma(\mu)}{\mu} = \alpha + i\beta.$$

Then there exists a constant C such that for all $f \in L_2(0,\infty)$,

$$\|kf\|_{L_2(0,1)} \leq C(\|e^{Re\gamma(\cdot)} f(\cdot)\|_{L_2(R^+)} + \|f(\cdot)\|_{L_2(R^+)})$$

Thus we can give the following Lemma 2.4, which can be proved easily by using the definition of the vector function space and the corresponding norm.

Lemma 2.4 *For any $\overrightarrow{F} \in L_2(0,\infty) \times L_2(0,\infty)$,let $K\overrightarrow{F}$ be the function defined by*

$$k\overrightarrow{F}(x) = \int_0^{+\infty} e^{\gamma(\mu)x} \overrightarrow{F}(\mu) d\mu$$

where $\gamma(\mu)$ is a continuous complex-valued function defined on $(0,\infty)$ satisfying the following two conditions:

(i)There exist $\delta > 0, b > 0$ such that

$$\sup_{0<\mu<\delta} \frac{|Re\gamma(\mu)|}{\mu} \geq b.$$

(ii)There exist a complex number $\alpha + i\beta$ such that

$$\lim_{\mu\to\infty} \frac{\gamma(\mu)}{\mu} = \alpha + i\beta.$$

Then there exists a constant C such that for all $\overrightarrow{F} \in L_2(0,\infty) \times L_2(0,\infty)$,

$$\|k\overrightarrow{F}\|_{L_2(0,1)\times L_2(0,1)} \leq C(\|e^{Re\gamma(\cdot)}\overrightarrow{F}(\cdot)\|_{L_2(R^+)\times L_2(R^+)} + \|\overrightarrow{F}(\cdot)\|_{L_2(R^+)\times L_2(R^+)})$$

We will give estimates for $\vec{w}_1, \vec{w}_2, \vec{w}_3$ in the following three lemmas. First, consider $\vec{w}_1(x,t) = \vec{I}_1(x,t) + I\vec{I}_1(x,t)$.

Lemma 2.5 *For any $S \in [0,3]$, there exist a constant C such that*

$$\|\vec{w_1}\|_{L_2(R^+;H^1(0,1))\times L_2(R^+;H^1(0,1))} + \sup_{0\le t<+\infty} \|\vec{w}_1(\cdot,t)\|_{L_2(0,1)\times L_2(0,1)} \le C\|\vec{h}_1\|_{H^{\frac{1}{3}}(R^+)\times H^{\frac{1}{3}}(R^+)} \tag{2.23}$$

and $\partial_x \vec{w}_1 \in C_b([0,1]; L_2(R^+)) \times C_b([0,1]; L_2(R^+))$ with

$$\sup_{x\in(0,1)} \|\partial_x \vec{w}_1(x,t)\|_{L_2(R^+)\times L_2(R^+)} \le C\|h_1\|_{H^{\frac{1}{3}}(R^+)\times H^{\frac{1}{3}}(R^+)} \tag{2.24}$$

for all $\vec{h}_1 \in H^{\frac{1}{3}}(R^+) \times H^{\frac{1}{3}}(R^+)$.

Proof. Since $\lambda_1(s) + \lambda_2(s) + \lambda_3(s) = 0$, we have

$$\triangle_{11,1}(s) = \triangle_{12,1}(s) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & e^{\lambda_2} & e^{\lambda_3} \\ 0 & \lambda_2 e^{\lambda_2} & \lambda_3 e^{\lambda_3} \end{pmatrix} = (\lambda_3 - \lambda_2)e^{-\lambda_1}$$

$$\triangle_{21,1}(s) = \triangle_{22,1}(s) = \begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1} & 0 & e^{\lambda_3} \\ \lambda_1 e^{\lambda_1} & 0 & \lambda_3 e^{\lambda_3} \end{pmatrix} = (\lambda_1 - \lambda_3)e^{-\lambda_2}$$

$$\triangle_{31,1}(s) = \triangle_{32,1}(s) = \begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1} & e^{\lambda_2} & 0 \\ \lambda_1 e^{\lambda_1} & \lambda_2 e^{\lambda_2} & 0 \end{pmatrix} = (\lambda_2 - \lambda_1)e^{-\lambda_3}$$

$$\triangle(s) = \begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1} & e^{\lambda_2} & e^{\lambda_3} \\ \lambda_1 e^{\lambda_1} & \lambda_2 e^{\lambda_2} & \lambda_3 e^{\lambda_3} \end{pmatrix} = (\lambda_3 - \lambda_2)e^{-\lambda_1} + (\lambda_1 - \lambda_3)e^{-\lambda_2} + (\lambda_2 - \lambda_1)e^{-\lambda_3}$$

we know $s = i\rho^3$,so

$$\frac{\Delta^+_{11,1}(\rho)}{\Delta(\rho)} \sim e^{-\frac{\sqrt{3}}{2}\rho}, \frac{\Delta^+_{21,1}(\rho)}{\Delta(\rho)} \sim e^{-\sqrt{3}\rho}, \frac{\Delta^+_{31,1}(\rho)}{\Delta(\rho)} \sim 1,$$

$$\frac{\Delta^+_{12,1}(\rho)}{\Delta(\rho)} \sim e^{-\frac{\sqrt{3}}{2}\rho}, \frac{\Delta^+_{22,1}(\rho)}{\Delta(\rho)} \sim e^{-\sqrt{3}\rho}, \frac{\Delta^+_{32,1}(\rho)}{\Delta(\rho)} \sim 1,$$

as $\rho \to +\infty$. Then we can estimate the solution $\vec{w}_1(x,t) = \vec{I}_1(x,t) + I\vec{I}_1(x,t)$.

$$\vec{I}_1 = (\sum_{j=1}^{3} \frac{1}{2\pi} \int_1^{+\infty} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta^+_{j1,1}(\rho)}{\Delta^+(\rho)} 3\rho^2 \hat{h}^+_{11}(\rho) d\rho, \sum_{j=1}^{3} \frac{1}{2\pi} \int_1^{+\infty} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta^+_{j2,1}(\rho)}{\Delta^+(\rho)} 3\rho^2 \hat{h}^+_{12}(\rho) d\rho)^T$$

by using Lemma 2.4, we obtain

$$\|\vec{I}_1(\cdot,t)\|^2_{L_2(0,1)\times L_2(0,1)} \leq C_1\sum_{j=1}^{3}\int_1^{+\infty}|\frac{\Delta^+_{j1,1}(\rho)}{\Delta^+(\rho)}|^2|\hat{h}^+_{11}(\rho)3\rho^2|^2d\rho+C_2\sum_{j=1}^{3}\int_1^{+\infty}|\frac{\Delta^+_{j2,1}(\rho)}{\Delta^+(\rho)}|^2|\hat{h}^+_{12}(\rho)3\rho^2|^2d\rho$$

let $C=max\{|\frac{\Delta^+_{j1,1}(\rho)}{\Delta^+(\rho)}|^2,|\frac{\Delta^+_{j2,1}(\rho)}{\Delta^+(\rho)}|^2\}, \rho\in[1,\infty), j=1,2,3; \mu=\rho^3$, we get

$$\begin{aligned}\|\vec{I}_1(\cdot,t)\|^2_{L_2(0,1)\times L_2(0,1)} &\leq C\int_0^{\infty}3\rho^2(|\hat{h}^+_{11}(\rho)|^2+|\hat{h}^+_{12}(\rho)|^2)3\rho^2)d\rho\\ = C\int_0^{\infty}(3\rho^2|\widehat{\vec{h}}^+_1(\rho)|^2 3\rho^2)d\rho &\leq C\int_0^{\infty}\mu^{\frac{2}{3}}|\int_0^{\infty}e^{-i\mu}\vec{h}_1(\tau)d\tau|^2d\mu\leq C\|\vec{h}_1\|_{H^{\frac{1}{3}}(R^+)\times H^{\frac{1}{3}}(R^+)}\end{aligned}$$

$t\geq 0$, we can obtain the estimate of $\overrightarrow{II_1}(x,t)$ similar to that appearing in the proof of $\vec{I_1}(x,t)$.

$$\|\vec{II}_1(\cdot,t)\|_{L_2(0,1)\times L_2(0,1)}\leq C\|\vec{h}_1\|_{H^{\frac{1}{3}}(R^+)\times H^{\frac{1}{3}}(R^+)}, t\geq 0.$$

Thus (2.23) holds. Then for $\partial_x\vec{I}_1(x,t)$, Let $\theta(\mu)$ be the real solution of $\mu=\rho^3, \rho\in[1,\infty)$, we have

$$\begin{aligned}\partial_x\vec{I}_1(x,t) &= (\sum_{j=1}^{3}\frac{1}{2\pi}\int_1^{+\infty}e^{i\rho^3t}(\lambda^+_j(\rho))e^{\lambda^+_j(\rho)x}\frac{\Delta^+_{j1,1}(\rho)}{\Delta^+(\rho)}3\rho^2\hat{h}^+_{11}(\rho)d\rho,\\ &\qquad \sum_{j=1}^{3}\frac{1}{2\pi}\int_1^{+\infty}e^{i\rho^3t}(\lambda^+_j(\rho))e^{\lambda^+_j(\rho)x}\frac{\Delta^+_{j2,1}(\rho)}{\Delta^+(\rho)}3\rho^2\hat{h}^+_{12}(\rho)d\rho)^T\\ &= (\sum_{i=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}e^{i\mu t}\lambda^+_j(\theta(\mu))e^{\lambda^+_j(\theta(\mu))x}\frac{\Delta^+_{j1,1}(\theta(\mu))}{\Delta^+(\theta(\mu))}\hat{h}_{11}(i\mu)d\mu,\\ &\qquad \sum_{i=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}e^{i\mu t}\lambda^+_j(\theta(\mu))e^{\lambda^+_j(\theta(\mu))x}\frac{\Delta^+_{j2,1}(\theta(\mu))}{\Delta^+(\theta(\mu))}\hat{h}_{12}(i\mu)d\mu)^T\end{aligned}$$

By using the Plancherel Theorem with respect to t, we deduce

$$\begin{aligned}\|\partial_x\vec{I}_1(x,\cdot)\|^2_{L_2(R^+)\times L_2(R^+)} \leq &\sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}|\lambda^+_j(\theta(\mu))e^{\lambda^+_j(\theta(\mu))x}\frac{\Delta^+_{j1,1}(\theta(\mu))}{\Delta^+(\theta(\mu))}|^2|\hat{h}_{11}(i\mu)|^2d\mu\\ &+\sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}|\lambda^+_j(\theta(\mu))e^{\lambda^+_j(\theta(\mu))x}\frac{\Delta^+_{j2,1}(\theta(\mu))}{\Delta^+(\theta(\mu))}|^2|\hat{h}_{12}(i\mu)|^2d\mu\end{aligned}$$

then

$$\begin{aligned}&\int_0^1\|\partial_x\vec{I}_1(x,\cdot)\|^2_{L_2(R^+)\times L_2(R^+)}dx\leq\sup_{x\in[0,1)}\|\partial_x\vec{I}_1(x,t)\|^2_{L_2(R^+)\times L_2(R^+)}\\ \leq\ & C\sum_{j=1}^{3}\int_0^{+\infty}|\lambda^+_j(\theta(\mu))|^2\sup_{x\in(0,1)}|e^{\lambda^+_j(\theta(\mu))x}|^2|\frac{\Delta^+_{j1,1}(\theta(\mu))}{\Delta^+(\theta(\mu))}|^2|\hat{h}_{11}(i\mu)|^2d\mu\end{aligned}$$

$$
\begin{aligned}
&+C\sum_{j=1}^{3}\int_{0}^{+\infty}|\lambda_j^+(\theta(\mu))|^2\sup_{x\in(0,1)}|e^{\lambda_j^+(\theta(\mu))x}|^2|\frac{\Delta_{j2,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}|^2|\hat{h}_{12}(i\mu)|^2d\mu \\
\leq\ & C\sum_{j=1}^{3}\int_{0}^{+\infty}(1+\mu)^{\frac{2}{3}}|\hat{h}_{11}(i\mu)|^2d\mu+C\sum_{j=1}^{3}\int_{0}^{+\infty}(1+\mu)^{\frac{2}{3}}|\hat{h}_{12}(i\mu)|^2d\mu \\
\leq\ & C\|\vec{h}_1\|_{H^{\frac{1}{3}}(R^+)\times H^{\frac{1}{3}}(R^+)}
\end{aligned}
$$

Next, we give the continuity of $\partial_x\vec{I}_1$ from$[0,1]\times[0,1]$ to $L_2(R^+)\times L_2(R^+)$, for any $x_0\in[0,1], x\in[0,1]$ we have

$$
\begin{aligned}
&\|\partial_x\vec{I}_1(x,\cdot)-\partial_x\vec{I}_1(x_0,\cdot)\|^2_{L_2(R^+)\times L_2(R^+)} \\
\leq\ & \sum_{j=1}^{3}\frac{1}{2\pi}\int_{0}^{+\infty}|\lambda_j^+(\theta(\mu))(e^{\lambda_j^+(\theta(\mu))x}-e^{\lambda_j^+(\theta(\mu))x_0})\frac{\Delta_{j1,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}|^2|\hat{h}_{11}(i\mu)|^2d\mu \\
&+\sum_{j=1}^{3}\frac{1}{2\pi}\int_{0}^{+\infty}|\lambda_j^+(\theta(\mu))(e^{\lambda_j^+(\theta(\mu))x}-e^{\lambda_j^+(\theta(\mu))x_0})\frac{\Delta_{j2,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}|^2|\hat{h}_{12}(i\mu)|^2d\mu \\
\leq\ & C\sum_{j=1}^{3}\frac{1}{2\pi}\int_{0}^{+\infty}(1+\mu)^{\frac{2}{3}}(|\hat{h}_{11}(i\mu)|^2+|\hat{h}_{12}(i\mu)|^2)d\mu \\
=\ & C\sum_{j=1}^{3}\frac{1}{2\pi}\int_{0}^{+\infty}(1+\mu)^{\frac{2}{3}}|\widehat{\vec{h}}_1(i\mu)|^2d\mu
\end{aligned}
$$

let $x\to x_0$, by using the Fatou's Lemma, we have $\|\partial_x\vec{I}_1(x,\cdot)-\partial_x\vec{I}_1(x_0,\cdot)\|^2_{L_2(R^+)\times L_2(R^+)}\to 0$, thus $\vec{I}_1(x,\cdot)\in C_b([0,1];L_2(R^+))\times C_b([0,1];L_2(R^+))$. Similar to that appearing in the proof of $\partial_x\vec{I}_1(x,t)$, we get

$$
\int_0^1\|\partial_x\vec{II}_1(x,\cdot)\|^2_{L_2(R^+)\times L_2(R^+)}dx\leq\sup_{x\in(0,1)}\|\partial_x\vec{II}_1(x,\cdot)\|^2_{L_2(R^+)\times L_2(R^+)}\leq C\|\vec{h}_1\|_{H^{\frac{1}{3}}(R^+)\times H^{\frac{1}{3}}(R^+)}
$$

and $\vec{II}_1(x,\cdot)\in C_b([0,1];L_2(R^+))\times C_b([0,1];L_2(R^+))$. This complete the proof. $\square$

The properties of $\vec{w_2}(x,t),\vec{w_3}(x,t)$ is similar to $\vec{w_1}(x,t)$ in Lemma 2.5, and the proof of $\vec{w_2}(x,t),\vec{w_3}(x,t)$ is same as that appearing in the Lemma 2.5, we omit it. We note that

$$
\frac{\Delta_{11,2}^+(\rho)}{\Delta(\rho)}\sim 1,\qquad \frac{\Delta_{12,2}^+(\rho)}{\Delta(\rho)}\sim 1,\qquad \frac{\Delta_{21,2}^+(\rho)}{\Delta(\rho)}\sim e^{-\frac{\sqrt{3}}{2}\rho},
$$

$$
\frac{\Delta_{22,2}^+(\rho)}{\Delta(\rho)}\sim e^{-\frac{\sqrt{3}}{2}\rho},\qquad \frac{\Delta_{31,2}^+(\rho)}{\Delta(\rho)}\sim 1,\qquad \frac{\Delta_{32,2}^+(\rho)}{\Delta(\rho)}\sim 1
$$

$$\frac{\Delta^+_{11,3}(\rho)}{\Delta(\rho)} \sim \rho^{-1}, \qquad \frac{\Delta^+_{12,3}(\rho)}{\Delta(\rho)} \sim \rho^{-1}, \qquad \frac{\Delta^+_{21,3}(\rho)}{\Delta(\rho)} \sim \rho^{-1}e^{-\frac{\sqrt{3}}{2}\rho}$$

$$\Delta^+_{22,3}(\rho)\Delta(\rho) \sim \rho^{-1}e^{-\frac{\sqrt{3}}{2}\rho}, \qquad \frac{\Delta^+_{31,2}(\rho)}{\Delta(\rho)} \sim \rho^{-1}, \qquad \frac{\Delta^+_{32,2}(\rho)}{\Delta(\rho)} \sim \rho^{-1}$$

We have the following Lemmas:

Lemma 2.6 *There exist a constant C such that*

$$\begin{aligned}
&\|\vec{w}_2\|_{L_2(R^+;H^1(0,1))\times L_2(R^+;H^1(0,1))} + \sup_{0\le t<+\infty} \|\vec{w}_2(\cdot,t)\|_{L_2(0,1)\times L_2(0,1)} \\
\le\ & C\|\vec{h}_2\|_{H^{\frac{1}{3}}(R^+)\times H^{\frac{1}{3}}(R^+)},
\end{aligned} \tag{2.25}$$

and $\partial_x\vec{w}_2 \in C_b([0,1];L_2(R^+)) \times C_b([0,1];L_2(R^+))$ with

$$\sup_{x\in(0,1)} \|\partial_x\vec{w}_2(x,t)\|_{L_2(R^+)\times L_2(R^+)} \le C\|h_2\|_{H^{\frac{1}{3}}(R^+)\times H^{\frac{1}{3}}(R^+)} \tag{2.26}$$

for all $\vec{h}_2 \in H^{\frac{1}{3}}(R^+) \times H^{\frac{1}{3}}(R^+)$.

Lemma 2.7 *There exist a constant C such that*

$$\begin{aligned}
&\|\vec{w}_3\|_{L_2(R^+;H^1(0,1))\times L_2(R^+;H^1(0,1))} + \sup_{0\le t<+\infty} \|\vec{w}_3(\cdot,t)\|_{L_2(0,1)\times L_2(0,1)} \\
\le\ & C\|\vec{h}_3\|_{L_2(R^+)\times L_2(R^+)},
\end{aligned} \tag{2.27}$$

and $\partial_x\vec{w}_3 \in C_b([0,1];L_2(R^+)) \times C_b([0,1];L_2(R^+))$ with

$$\sup_{x\in(0,1)} \|\partial_x\vec{w}_2(x,t)\|_{L_2(R^+)\times L_2(R^+)} \le C\|h_2\|_{H^{\frac{1}{3}}(R^+)\times H^{\frac{1}{3}}(R^+)} \tag{2.28}$$

for all $\vec{h}_3 \in L_2(R^+) \times L_2(R^+)$.

We write the solution $\vec{w}$ of (2.16)-(2.17) as

$$\vec{w}(t) = \sum_{j=1}^{3} Q_j(t)\vec{h}_j = Q_b(t)H$$

where the spatial variable x is suppressed and Q_j are as defined in (2.22). For $s \ge 0, T > 0$, let

$$\mathcal{H}_{s,T} = H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$$

For any $H \in \mathcal{H}_{s,T} \times \mathcal{H}_{s,T}$, we give

$$\|H\|^2_{\mathcal{H}_{s,T}\times\mathcal{H}_{s,T}} = \|\vec{h_1}\|^2_{H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s+1}{3}}(0,T)} + \|\vec{h_2}\|^2_{H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s+1}{3}}(0,T)} + \|\vec{h_3}\|^2_{H^{\frac{s}{3}}(0,T)\times H^{\frac{s}{3}}(0,T)}.$$

Then we have the following theorem:

Theorem 2.8 *Assume that $H \in \mathcal{H}_{0,\infty} \times \mathcal{H}_{0,\infty}$, the problem (2.16)-(2.18) admits a unique solution*

$$\vec{w}(x,t) = [Q_b(t)H](x)$$

which belongs to the space $(C_b(R^+; L_2(0,1)) \cap L_2(R^+; H^1(0,1)) \times (C_b(R^+; L_2(0,1)) \cap L_2(R^+; H^1(0,1)))$ with

$$\|\vec{w}\|_{L_2(R^+;H^1(0,1))\times L_2(R^+;H^1(0,1))} + sup_{t\in[0,+\infty)}\|\vec{w}(\cdot,t)\|_{L_2(0,1)} \le C\|H\|_{\mathcal{H}_{0,\infty}\times\mathcal{H}_{0,\infty}}$$

and $\partial_x\vec{w} \in C_b([0,1]; L_2(R^+)) \times C_b([0,1]; L_2(R^+))$ with

$$sup_{x\in(0,1)}\|\partial_x\vec{w}(x,\cdot)\|_{L_2(R^+)\times L_2(R^+)} \le C\|H\|_{\mathcal{H}_{0,\infty}\times\mathcal{H}_{0,\infty}}$$

where C is a constant.

3 Local well-posedness

In this section, the local well-posedness of the nonlinear IBVP will be discussed.

$$\vec{w}_t + \vec{w}_{xxx} + B\vec{w} = \vec{0} \tag{3.1}$$

$$\vec{w}(x,0) = \vec{\phi}(x) \tag{3.2}$$

$$\vec{w}(0,t) = \vec{h}_1(t), \vec{w}(1,t) = \vec{h}_2(t), \vec{w}_x(1,t) = \vec{h}_3(t) \tag{3.3}$$

where $\vec{w} = (u,v)^T, \vec{\phi} = (u_0(x), v_0(x))^T, \vec{h}_i(t) = (h_{i1}(t), h_{i2}(t))^T, B = \begin{pmatrix} 6u_x & -2bv_x \\ 3v_x & 0 \end{pmatrix}$. For any $T > 0, s \ge 0$,let

$$X_{s,T} = H^s(0,1) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$$

$$Y_{s,T} = \ \{\vec{w} : \vec{w} \in (C([0,T]; H^s(0,1)) \cap L_2([0,T]; H^{s+1}(0,1))) \times (C([0,T]; H^s(0,1))$$

$$\cap L_2([0,T];H^{s+1}(0,1))), \vec{w}_x \in (C([0,1];L_2(0,T)) \times C([0,1];L_2(0,T)))\}$$

define the norm $\|\cdot\|_{Y_{s,T}}$ as:

$$\|\vec{w}\|_{Y_{s,T}} = (\|\vec{w}\|^2_{C([0,T];H^s(0,1))} + \|\vec{w}\|^2_{L_2([0,T];H^{s+1}(0,1))} + \|\vec{w}_x\|^2_{C([0,1];L_2(0,T))})^{\frac{1}{2}},$$

$\vec{w} \in Y_{s,T}$, then we have the following lemma:

Lemma 3.1 *For any $s \geq 0, T \geq 0$ and $\vec{w} \in Y_{s,T}$, we have*

$$\|B\vec{w}\|_{L_1([0,T];H^s(0,1))\times L_1([0,T];H^s(0,1))} \leq C(T^{\frac{1}{2}} + T^{\frac{1}{3}})\|\vec{w}\|^2_{Y_{s,T}} \tag{3.4}$$

where C is a constant, $B = \begin{pmatrix} 6u_x & -2bv_x \\ 3v_x & 0 \end{pmatrix}$ *, b is a real positive constant.*

Proof. We will give the proof for $0 \leq s \leq 1$, and we can get the proof of other values of s by similar method. First, we consider $s = 0$:

$$\begin{aligned}
& \|B\vec{w}\|_{L_1([0,T];L_2(0,1))\times L_1([0,T];L_2(0,1))} \\
\leq\ & 6\int_0^T \|u(\cdot,t)u_x(\cdot,t)\|_{L_2(0,1)}dt + 2b\int_0^T \|v(\cdot,t)v_x(\cdot,t)\|_{L_2(0,1)}dt + 3\int_0^T \|u(\cdot,t)v_x(\cdot,t)\|_{L_2(0,1)}dt
\end{aligned}$$

by applying the Poincare inequality, we have

$$\begin{aligned}
& \|u(\cdot,t)v_x(\cdot,t)\|_{L_2(0,1)} \leq \|u(\cdot,t)\|_{L_\infty(0,1)}\|v_x(\cdot,t)\|_{L_2(0,1)} \\
\leq\ & C(\|u(\cdot,t)\|_{L_2(0,1)} + \|u(\cdot,t)\|^{\frac{1}{2}}_{L_2(0,1)}\|u_x(\cdot,t)\|^{\frac{1}{2}}_{L_2(0,1)})\|v_x(\cdot,t)\|_{L_2(0,1)},
\end{aligned}$$

then by integrating these two terms with respect to t, we have

$$\begin{aligned}
& \int_0^T \|u(\cdot,t)\|_{L_2(0,1)}\|v_x(\cdot,t)\|_{L_2(0,1)}dt \leq sup_{0\leq t\leq T}\|u(\cdot,t)\|_{L_2(0,1)}\int_0^T \|v_x(\cdot,t)\|_{L_2(0,1)}dt \\
& \leq T^{\frac{1}{2}} sup_{0\leq t\leq T}\|u(\cdot,t)\|_{L_2(0,1)}(\int_0^T \|v_x(\cdot,t)\|^2_{L_2(0,1)}dt)^{\frac{1}{2}} \leq CT^{\frac{1}{2}}\|\vec{w}\|^2_{Y_{0,T}},
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^T \|u(\cdot,t)\|^{\frac{1}{2}}_{L_2(0,1)}\|u_x(\cdot,t)\|^{\frac{1}{2}}_{L_2(0,1)}\|v_x(\cdot,t)\|_{L_2(0,1)}dt \\
\leq\ & sup_{0\leq t\leq T}\|u(\cdot,t)\|^{\frac{1}{2}}_{L_2(0,1)}(\int_0^T \|u_x(\cdot,t)\|^2_{L_2(0,1)}dt)^{\frac{1}{4}}(\int_0^T \|v_x(\cdot,t)\|^{\frac{4}{3}}_{L_2(0,1)}dt)^{\frac{3}{4}} \\
\leq\ & CT^{\frac{1}{3}}\|\vec{w}\|^2_{Y_{0,T}}.
\end{aligned}$$

Then

$$\int_0^T \|u(\cdot,t)v_x(\cdot,t)\|_{L_2(0,1)}dt \le C(T^{\frac{1}{2}}+T^{\frac{1}{3}})\|\vec{w}\|^2_{Y_{0,T}},$$

similarly

$$\int_0^T \|u(\cdot,t)u_x(\cdot,t)\|_{L_2(0,1)}dt \le C(T^{\frac{1}{2}}+T^{\frac{1}{3}})\|\vec{w}\|^2_{Y_{0,T}},$$

$$\int_0^T \|v(\cdot,t)v_x(\cdot,t)\|_{L_2(0,1)}dt \le C(T^{\frac{1}{2}}+T^{\frac{1}{3}})\|\vec{w}\|^2_{Y_{0,T}}.$$

Then,

$$\|B\vec{w}\|_{L_1([0,T];L_2(0,1))\times L_1([0,T];L_2(0,1))} \le C(T^{\frac{1}{2}}+T^{\frac{1}{3}})\|\vec{w}\|^2_{Y_{0,T}}.$$

For $s=1$, we note that

$$\begin{aligned}\|u(\cdot,t)v_x(\cdot,t)\|_{H^1(0,1)} &\le \|u(\cdot,t)v_x(\cdot,t)\|_{L_2(0,1)}+\|u_x(\cdot,t)v_x(\cdot,t)\|_{L_2(0,1)}+\|u(\cdot,t)v_{xx}(\cdot,t)\|_{L_2(0,1)}\\ &\le C(T^{\frac{1}{2}}+T^{\frac{1}{3}})(\|\vec{w}\|^2_{Y_{0,T}}+\|\vec{w}_x\|^2_{Y_{0,T}}+\|\vec{w}_x\|^2_{Y_{0,T}})\\ &\le C(T^{\frac{1}{2}}+T^{\frac{1}{3}})\|\vec{w}\|^2_{Y_{1,T}}\end{aligned}$$

similarly

$$\int_0^T \|u(\cdot,t)u_x(\cdot,t)\|_{H^1(0,1)}dt \le C(T^{\frac{1}{2}}+T^{\frac{1}{3}})\|\vec{w}\|^2_{Y_{1,T}},$$

$$\int_0^T \|v(\cdot,t)v_x(\cdot,t)\|_{H^1(0,1)}dt \le C(T^{\frac{1}{2}}+T^{\frac{1}{3}})\|\vec{w}\|^2_{Y_{1,T}}.$$

Then

$$\begin{aligned}&\|B\vec{w}\|_{L_1([0,T];H^1(0,1))\times L_1([0,T];H^1(0,1))}\\ \le\ & 6\int_0^T \|u(\cdot,t)u_x(\cdot,t)\|_{H^1(0,1)}dt+2b\int_0^T \|v(\cdot,t)v_x(\cdot,t)\|_{H^1(0,1)}dt+3\int_0^T \|u(\cdot,t)v_x(\cdot,t)\|_{H^1(0,1)}dt\\ \le\ & C(T^{\frac{1}{2}}+T^{\frac{1}{3}})\|\vec{w}\|^2_{Y_{1,T}}\end{aligned}$$

Next, we will use the nonlinear interpolation theory to give the estimate (3.4) with $0 < s < 1$ [3]. Let B_0 and B_1 be two Banach space such that $B_1 \subset B_0$ with the inclusion map continuous. Let $\vec{F} \in B_0 \times B_0, \vec{F} = (f_1, f_2)^T, \vec{G} = (g_1, g_2)^T$, define

$$K(f_i,\epsilon) = inf_{g_i\in B_1}\{\|f_i-g_i\|_{B_0}+\epsilon\|g_i\|_{B_1}\}, K(\vec{F},\epsilon) = K(f_1,\epsilon)+K(f_2,\epsilon),$$

where $\epsilon \geq 0, i = 1, 2$. For $\theta \in (0,1), p \in [1, +\infty]$, define

$$[B_0, B_1]_{\theta,p} = B_{\theta,p} = \{f_i \in B_0 : \|f_i\|_{B_{\theta,p}} = (\int_0^\infty K(f_i, \epsilon)^p \epsilon^{-\theta p - 1} dt)^{\frac{1}{p}} < +\infty\}$$

with the usual modification for the case $p = +\infty$. Then $B_{\theta,p}$ is a Banach space with the normal $\|\cdot\|_{B_{\theta,p}}$. Given two pairs of indices (θ_1, p_1) and (θ_2, p_2) as above, then we have $(\theta_1, p_1) < (\theta_1, p_1)$, which means $\begin{cases} \theta_1 < \theta_2 \\ \theta_1 = \theta_2, p_1 > p_2 \end{cases}$
If $(\theta_1, p_1) < (\theta_1, p_1)$ then $B_{\theta_1,p_1} \supset B_{\theta_2,p_2}$ with the inclusion map continuous. The next two Lemmas are given by Bona and Scott [3].

Lemma 3.2 *(Bona and Scott,1976) Let $f \in B_0, 0 < \theta < 1$ and $1 \leq p \leq \infty$. Suppose that for all $\epsilon > 0$ there are $g_i(\epsilon) \in B_i$ such that $f = g_0(\epsilon) + g_1(\epsilon)$ with $\|g_i(\epsilon)\|_{B_i} \leq G_i(\epsilon)$ and such that $M_i = (\int_0^\infty G_i(\epsilon)^p \epsilon^{(i-\theta)p-1} d\epsilon)^{\frac{1}{p}} < +\infty$. For $i = 0, 1$. Then $f \in B_{\theta,p}$ and $\|f\|_{B_{\theta,p}} \leq M_0^{1-\theta} M_1^\theta$*

Lemma 3.3 *(Bona and Scott,1976) Let $f \in B_0, f_\epsilon \in B_1$ satisfy the inequality*

$$\|f - f_\epsilon\|_{B_0} + \epsilon \|f_\epsilon\|_{B_1} \leq 2K(f, \epsilon)$$

for some $\epsilon > 0$. If $f \in B_{\theta,p}$ for some θ and p with $0 < \theta < 1$ and $1 \leq p \leq \infty$, then

$$\|f_\epsilon\|_{B_{\theta,p}} \leq 3\|f\|_{B_{\theta,p}}.$$

Then we can give the following Lemmas, which can be proved easily by using the definition of the vector function space and the corresponding norm.

Lemma 3.4 *Let $\vec{F} \in B_0 \times B_0, 0 < \theta < 1$ and $1 \leq p \leq \infty$. Suppose that for all $\epsilon > 0$ there are $\vec{G}_i(\epsilon) \in B_i \times B_i$ such that $\vec{F} = \vec{G}_0(\epsilon) + \vec{G}_1(\epsilon)$ with $\|\vec{G}_i(\epsilon)\|_{B_i \times B_i} \leq Q_i(\epsilon)$ and such that $M_i = (\int_0^\infty Q_i(\epsilon)^p \epsilon^{(i-\theta)p-1} d\epsilon)^{\frac{1}{p}} < +\infty$. For $i = 0, 1$. Then $\vec{F} \in B_{\theta,p} \times B_{\theta,p}$ and $\|\vec{F}\|_{B_{\theta,p} \times B_{\theta,p}} \leq M_0^{1-\theta} M_1^\theta$*

Lemma 3.5 *Let $\vec{F} \in B_0 \times B_0, \vec{F}_\epsilon \in B_1 \times B_1$ satisfy the inequality*

$$\|\vec{F} - \vec{F}_\epsilon\|_{B_0 \times B_0} + \epsilon \|\vec{F}_\epsilon\|_{B_1 \times B_1} \leq 2K(\vec{F}, \epsilon)$$

for some $\epsilon > 0$. If $\vec{F} \in B_{\theta,p} \times B_{\theta,p}$ for some θ and p with $0 < \theta < 1$ and $1 \leq p \leq \infty$, then

$$\|\vec{F}_\epsilon\|_{B_{\theta,p} \times B_{\theta,p}} \leq 3\|\vec{F}\|_{B_{\theta,p} \times B_{\theta,p}}.$$

With these preliminaries in hand, the abstract result on boundedness of mappings of intermediate spaces can be established.

Lemma 3.6 *Let B_0^j and B_1^j be Banach spaces such that $B_0^j \supset B_1^j$ with continuous inclusion mappings, $j = 1, 2$. Let λ and q lie in the ranges $0 < \lambda < 1$ and $1 \leq q \leq +\infty$. Suppose A is a mapping such that*

(i).$A : B_{\lambda,q}^1 \times B_{\lambda,q}^1 \to B_0^2 \times B_0^2$ and for $\vec{F} = (f_1, f_2), \vec{G} = (g_1, g_2) \in B_{\lambda,q}^1 \times B_{\lambda,q}^1$,

$$\|A\vec{F} - A\vec{G}\|_{B_0^2 \times B_0^2} \leq C_0(\|\vec{F}\|_{B_{\lambda,q}^1 \times B_{\lambda,q}^1} + \|\vec{G}\|_{B_{\lambda,q}^1 \times B_{\lambda,q}^1})\|\vec{F} - \vec{G}\|_{B_0^1 \times B_0^1}$$

and

(ii).$A : B_1^1 \times B_1^1 \to B_1^2 \times B_1^2$ and for $\vec{H} = (h_1, h_2) \in B_1^1 \times B_1^1$,

$$\|A\vec{H}\|_{B_1^2 \times B_1^2} \leq C_1(\|\vec{H}\|_{B_{\lambda,q}^1 \times B_{\lambda,q}^1})\|\vec{H}\|_{B_1^1 \times B_1^1},$$

where C_j are continuous nondecreasing functions, $j = 0, 1$. Then if $(\theta, p) \geq (\lambda, q)$, A maps $B_{\theta,p}^1 \times B_{\theta,p}^1$ into $B_{\theta,p}^2 \times B_{\theta,p}^2$ and for $\vec{F} \in B_{\theta,p}^1 \times B_{\theta,p}^1$,

$$\|A\vec{F}\|_{B_{\theta,p}^2 \times B_{\theta,p}^2} \leq C(\|\vec{F}\|_{B_{\lambda,q}^1 \times B_{\lambda,q}^1}))\|\vec{F}\|_{B_{\theta,p}^1 \times B_{\theta,p}^1})$$

Proof. Let $\vec{F} \in B_{\theta,p}^1 \times B_{\theta,p}^1$ and for each $\epsilon > 0$, choose $\vec{F}_\epsilon \in B_1^1 \times B_1^1$ such that

$$\|\vec{F} - \vec{F}_\epsilon\|_{B_0^1 \times B_0^1} + \epsilon\|\vec{F}\|_{B_1^1 \times B_1^1} \leq 2K(\vec{F}, \epsilon).$$

Let $(\theta, p) \geq (\lambda, q)$, then $B_{\theta,p}^1 \subset B_{\lambda,q}^1$, we have $\vec{F} \in B_{\lambda,q}^1$. From Lemma 3.5, we get $\|\vec{F}_\epsilon\|_{B_{\lambda,q}^1 \times B_{\lambda,q}^1} \leq 3\|\vec{F}\|_{B_{\lambda,q}^1 \times B_{\lambda,q}^1}$. Hypotheses (i) and (ii) therefore yield

$$\begin{aligned} \|A\vec{F} - A\vec{F}_\epsilon\|_{B_0^2 \times B_0^2} &\leq C_0(\|\vec{F}\|_{B_{\lambda,q}^1 \times B_{\lambda,q}^1} + \|\vec{F}_\epsilon\|_{B_{\lambda,q}^1 \times B_{\lambda,q}^1})\|\vec{F} - \vec{F}_\epsilon\|_{B_0^1 \times B_0^1} \\ &\leq 2C_0(4\|\vec{F}\|_{B_{\lambda,q}^1 \times B_{\lambda,q}^1})K(\vec{F}, \epsilon) \end{aligned}$$

and

$$\begin{aligned} \epsilon\|A\vec{F}_\epsilon\|_{B_1^2 \times B_1^2} &\leq C_1(\|\vec{F}_\epsilon\|_{B_{\lambda,q}^1 \times B_{\lambda,q}^1})\epsilon\|\vec{F}_\epsilon\|_{B_1^1 \times B_1^1} \\ &\leq 2C_1(3\|\vec{F}\|_{B_{\lambda,q}^1 \times B_{\lambda,q}^1})K(\vec{F}, \epsilon). \end{aligned}$$

Let $\vec{G}_0(\epsilon) = A\vec{F} - A\vec{F}_\epsilon, \vec{G}_1(\epsilon) = A\vec{F}_\epsilon$, we get $A\vec{F} = \vec{G}_0(\epsilon) + \vec{G}_1(\epsilon)$, then

$$\begin{aligned} M_i &= 2C_i((4-i)\|\vec{F}\|_{B^1_{\lambda,q}\times B^1_{\lambda,q}})(\int_0^{+\infty} K(\vec{F},\delta)^p\delta^{-\theta p-1}d\delta)^{\frac{1}{p}} \\ &= 2C_i((4-i)\|\vec{F}\|_{B^1_{\lambda,q}\times B^1_{\lambda,q}})\|\vec{F}\|_{B^1_{\theta,p}\times B^1_{\theta,p}} \end{aligned}$$

$i = 0, 1$. From Lemma 3.4, we establish the stated conclusion. □

To prove that estimate(3.4) holds for $0 \le s \le 1$, choose $B^1_0 \times B^1_0 = Y_{0,T}, B^1_1 \times B^1_1 = Y_{3,T}, B^2_0\times B^2_0 = L_1(0,T;L_2(0,1))\times L_1(0,T;L_2(0,1)), B^2_1\times B^2_1 = L_1(0,T;H^1(0,1))\times L_1(0,T;H^1(0,1))$. For given $0 < s < 1$, choose $p = 2, \theta = s$. Then

$$B^2_{\theta,p} \times B^2_{\theta,p} = L_1(0,T;H^s(0,1)) \times L_1(0,T;H^s(0,1)), B^1_{\theta,p} \times B^1_{\theta,p} = Y_{s,T}.$$

In this case, assumption(ii) of Lemma 3.6 has been proved with $s = 1$. So we only need to verify assumption(i). Let $\vec{\omega}_1, \vec{\omega}_2$ satisfy (3.1)-(3.3), $\vec{\omega}_1 = (u_1, v_1)^T, \vec{\omega}_2 = (u_2, v_2)^T, B_1 = \begin{pmatrix} 6u_{1x} & -2bv_{1x} \\ 3v_{1x} & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 6u_{2x} & -2bv_{2x} \\ 3v_{2x} & 0 \end{pmatrix}$, then

$$\begin{aligned} &\|B_1\vec{\omega}_1 - B_2\vec{\omega}_2\|_{L_1(0,T;L_2(0,1))\times L_1(0,T;L_2(0,1))} \\ \le\ & 6\int_0^T \|u_{1x}(u_1-u_2)\|_{L_2(0,1)}dt + 6\int_0^T \|u_2(u_{1x}-u_{2x})\|_{L_2(0,1)}dt + 2b\int_0^T \|v_{1x}(v_1-v_2)\|_{L_2(0,1)}dt \\ & +2b\int_0^T \|v_2(v_{1x}-v_{2x})\|_{L_2(0,1)}dt + 3\int_0^T \|v_{1x}(u_1-u_2)\|_{L_2(0,1)}dt + 3\int_0^T \|u_2(v_{1x}-v_{2x})\|_{L_2(0,1)}dt, \end{aligned}$$

by the Poincare inequality, we have

$$\begin{aligned} &\|u_{1x}(\cdot,t)(u_1(\cdot,t)-u_2(\cdot,t))\|_{L_2(0,1)} \\ \le\ & \|u_1(\cdot,t)-u_2(\cdot,t)\|_{L_\infty(0,1)}\|u_{1x}(\cdot,t)\|_{L_2(0,1)} \\ \le\ & C(\|u_1(\cdot,t)-u_2(\cdot,t)\|_{L_2(0,1)} + \|u_1(\cdot,t)-u_2(\cdot,t)\|^{\frac{1}{2}}_{L_2(0,1)}\|u_{1x}(\cdot,t)-u_{2x}(\cdot,t)\|^{\frac{1}{2}}_{L_2(0,1)})\|u_{1x}(\cdot,t)\|_{L_2(0,1)} \end{aligned}$$

integrating these two terms with respect to t, we have

$$\begin{aligned} &\int_0^T \|u_1(\cdot,t)-u_2(\cdot,t)\|_{L_2(0,1)}\|u_{1x}(\cdot,t)\|_{L_2(0,1)}dt \\ \le\ & \sup_{0\le t\le T}\|u_1(\cdot,t)-u_2(\cdot,t)\|_{L_2(0,1)}\int_0^T \|u_{1x}(\cdot,t)\|_{L_2(0,1)}dt \\ \le\ & T^{\frac{1}{2}}sup_{0\le t\le T}\|u_1(\cdot,t)-u_2(\cdot,t)\|_{L_2(0,1)}(\int_0^T \|u_{1x}(\cdot,t)\|^2_{L_2(0,1)}dt)^{\frac{1}{2}} \end{aligned}$$

$$\leq \; CT^{\frac{1}{2}}\|\vec{\omega}_1-\vec{\omega}_2\|_{Y_{0,T}}(\|\vec{\omega}_1\|_{Y_{0,T}}+\|\vec{\omega}_2\|_{Y_{0,T}})$$

and

$$\int_0^T \|u_1(\cdot,t)-u_2(\cdot,t)\|^{\frac{1}{2}}_{L_2(0,1)}\|u_1(\cdot,t)-u_2(\cdot,t)\|^{\frac{1}{2}}_{L_2(0,1)}\|u_{1x}(\cdot,t)\|_{L_2(0,1)}dt$$
$$\leq sup_{0\leq t\leq T}\|u_1(\cdot,t)-u_2(\cdot,t)\|^{\frac{1}{2}}_{L_2(0,1)}(\int_0^T \|u_{1x}(\cdot,t)-u_{2x}(\cdot,t)\|^2_{L_2(0,1)}dt)^{\frac{1}{2}}(\int_0^T \|u_{1x}(\cdot,t)\|^{\frac{4}{3}}_{L_2(0,1)}dt)^{\frac{3}{4}}$$
$$\leq CT^{\frac{1}{3}}\|\vec{\omega}_1-\vec{\omega}_2\|_{Y_{0,T}}(\|\vec{\omega}_1\|_{Y_{0,T}}+\|\vec{\omega}_2\|_{Y_{0,T}}).$$

Other calculations in the same way, we can get

$$\|B_1\vec{\omega}_1-B_2\vec{\omega}_2\|_{L_1(0,T;L_2(0,1))\times L_1(0,T;L_2(0,1))}\leq C(T^{\frac{1}{2}}+T^{\frac{1}{3}})\|\vec{\omega}_1-\vec{\omega}_2\|_{Y_{0,T}}(\|\vec{\omega}_1\|_{Y_{0,T}}+\|\vec{\omega}_2\|_{Y_{0,T}}).$$

Thus assumption(i) of Lemma 3.6 is satisfied. Estimate(3.4) is established for $0<s<1$ by invoking Lemma 3.6. The proof of Lemma 3.1 is now complete. □

Next, we discuss that the IBVP(3.1)-(3.3) is locally well-posed on the space $X_{0,T}$.

Lemma 3.7 *Assume that $(\vec{\phi},\vec{h_1},\vec{h_2},\vec{h_3})\in X_{0,T}\times X_{0,T}, T>0$, then there exists a $T^*\in(0,T]$ such that the IBVP(3.1)-(3.3) admits a unique solution $\vec{w}\in Y_{0,T^*}$, where T^* relies on $\|(\vec{\phi},\vec{h_1},\vec{h_2},\vec{h_3})\|_{X_{0,T}\times X_{0,T}}$. Furthermore, for any $T'<T^*$, there exists a neighborhood U of $(\vec{\phi},\vec{h_1},\vec{h_2},\vec{h_3})$ such that the IBVP (3.1)-(3.3) admits a unique solution in the space $Y_{0,T'}$, where $(\vec{\psi},\vec{H_1},\vec{H_2},\vec{H_3})\in U$ and the corresponding solution map from U to $Y_{0,T'}$ is Lipschitz continuous.*

Proof. We covert (3.1)-(3.3) to an integral equation $(\vec{w}(t)\sim\vec{w}(x,t))$:

$$\vec{w}(t)=P_0(t)\vec{\phi}+Q_b(t)H-\int_0^t P_0(t-\tau)B\vec{w}(\tau)d\tau$$

where $Q_b(t)$ is given in Section 2. Assume that $(\vec{\phi},\vec{h_1},\vec{h_2},\vec{h_3})\in X_{0,T}\times X_{0,T}, S_{\theta,r}=\{\vec{w}\in Y_{0,\theta},\|\vec{w}\|_{Y_{0,\theta}}\leq r\}$, where $r>0,\theta>0$ are constants. Let $S_{\theta,r}$ is a closed, convex and bounded subset of the space $Y_{0,\theta}$, then $S_{\theta,r}$ is a complete metric space. We give a map Γ on $S_{\theta,r}$ as follows:

$$\Gamma(\vec{w})=P_0(t)\vec{\phi}+Q_b(t)H-\int_0^t P_0(t-\tau)B\vec{w}(\tau)d\tau, \vec{w}\in S_{\theta,r}$$

then we have

$$\begin{aligned}\|\Gamma(\vec{w})\|_{Y_{0,\theta}} &\leq C_0\|(\vec{\phi},\vec{h_1},\vec{h_2},\vec{h_3})\|_{X_{0,T}\times X_{0,T}} + \|B\vec{w}\|_{L_1(0,\theta;L_2(0,1))\times L_1(0,\theta;L_2(0,1))} \\ &\leq C_0\|(\vec{\phi},\vec{h_1},\vec{h_2},\vec{h_3})\|_{X_{0,T}\times X_{0,T}} + C_1(\theta^{\frac{1}{2}}+\theta^{\frac{1}{3}})\|\vec{w}\|^2_{Y_{0,\theta}}\end{aligned}$$

where C_0, C_1 are constants, choosing $r>0,\theta>0$ so that

$$r = 2C_0\|(\vec{\phi},\vec{h_1},\vec{h_2},\vec{h_3})\|_{X_{0,T}\times X_{0,T}}$$

$$C_1(\theta^{\frac{1}{2}}+\theta^{\frac{1}{3}})r \leq \frac{1}{2}$$

then

$$\|\Gamma(\vec{w})\|_{Y_{0,\theta}} \leq r, \vec{w}\in S_{\theta,r}$$

we can get

$$\|\Gamma(\vec{w}_1)-\Gamma(\vec{w}_2)\|_{Y_{0,\theta}} \leq \frac{1}{2}\|\vec{w}_1-\vec{w}_2\|_{Y_{0,\theta}}$$

where $\vec{w}_1,\vec{w}_2\in S_{\theta,r}$. Which means that the map Γ is a contraction mapping of $S_{\theta,r}$. And its fixed point $\vec{w}=\Gamma(\vec{w})$ is the unique solution of IBVP (1.1)-(1.5) in $S_{\theta,r}$. $\square$

The next lemma gives an estimate for solution of the following forced linear problem (3.5)-(3.7) in the space $Y_{s,T}$, where $0\leq s\leq 3$.

$$\vec{w}_t + \vec{w}_{xxx} = \vec{F} \tag{3.5}$$

$$\vec{w}(x,0) = \vec{\phi}(x) \tag{3.6}$$

$$\vec{w}(0,t) = \vec{h_1}(t), \vec{w}(1,t) = \vec{h_2}(t), \vec{w}_x(1,t) = \vec{h_3}(t) \tag{3.7}$$

Lemma 3.8 *Assume that $T>0, s\in[0,3], s\neq \frac{1}{2},\frac{3}{2}$, for given $\vec{F}\in W^{\frac{s}{3},1}([0,T];L_2(0,1))\times$ $W^{\frac{s}{3},1}([0,T];$*
$L_2(0,1)),(\vec{\phi},\vec{h_1},\vec{h_2},\vec{h_3})\in X_{s,T}\times X_{s,T}$ satisfying the compatibility conditions

$$\vec{\phi}(0) = \vec{h_1}(0), \vec{\phi}(1) = \vec{h_2}(0), (s\in(\frac{1}{2},\frac{3}{2}])$$

$$\vec{\phi}(0) = \vec{h_1}(0), \vec{\phi}(1) = \vec{h_2}(0), \vec{\phi}'(1) = \vec{h_3}(0), (s\in(\frac{3}{2},3])$$

then there exists a unique solution $\vec{w} \in Y_{s,T}$ *of* (3.5)-(3.7) *and*

$$\|\vec{w}\|_{Y_{s,T}} \leq C(\|(\vec{\phi}, \vec{h_1}, \vec{h_2}, \vec{h_3})\|_{X_{s,T}\times X_{s,T}} + \|\vec{F}\|_{W^{\frac{s}{3},1}([0,T];L_2(0,1))\times W^{\frac{s}{3},1}([0,T];L_2(0,1))}) \tag{3.8}$$

where C *is a positive constant. Furthermore, if* $s = 3$*, we have* $\vec{w}_t \in Y_{0,T}$ *and*

$$\|\vec{w}_t\|_{Y_{0,T}} \leq C(\|(\vec{\phi}, \vec{h_1}, \vec{h_2}, \vec{h_3})\|_{X_{3,T}\times X_{3,T}} + \|\vec{F}\|_{W^{1,1}([0,T];L_2(0,1))\times W^{1,1}([0,T];L_2(0,1))}).$$

Proof. Using the linear estimates obtained in Section 2, for $(\vec{\phi}, \vec{h_1}, \vec{h_2}, \vec{h_3}) \in X_{0,T} \times X_{0,T}$ and $\vec{F} \in L_1(0,T;L_2(0,1)) \times L_1(0,T;L_2(0,1))$, there exists a solution $\vec{w}$ of (3.5)-(3.7) satisfies $\vec{w} \in Y_{0,T}$ and

$$\|\vec{w}\|_{Y_{0,T}} \leq C(\|(\vec{\phi}, \vec{h_1}, \vec{h_2}, \vec{h_3})\|_{X_{0,T}\times X_{0,T}} + \|\vec{F}\|_{L_1(0,T;L_2(0,1))\times L_1(0,T;L_2(0,1))}) \tag{3.9}$$

where C is a constant. Let $\vec{z} = \vec{w}_t$, then the system (3.5)-(3.7) can be written as

$$\vec{z}_t + \vec{z}_{xxx} = \vec{F}_t \tag{3.10}$$

$$\vec{z}(x,0) = \vec{F}(x,0) - \vec{\phi}'''(x) \tag{3.11}$$

$$\vec{z}(0,t) = \vec{h}_1'(t), \vec{z}(1,t) = \vec{h}_2'(t), \vec{z}_x(1,t) = \vec{h}_3'(t) \tag{3.12}$$

from (3.9) we can get the solution $\vec{z}(x,t)$ of system (3.10)-(3.12) satisfies

$$\|\vec{z}\|_{Y_{0,T}} \leq C(\|\vec{F}_t\|_{L_1(0,T;L_2(0,1))\times L_1(0,T;L_2(0,1))} + \|(\vec{F}(\cdot,0) - \vec{\phi}'''(x), \vec{h}_1', \vec{h}_2', \vec{h}_3')\|_{X_{0,T}\times X_{0,T}})$$

let

$$\vec{w}(x,t) = \int_0^t \vec{z}(x,\tau)d\tau + \vec{\phi}(x)$$

then we get

$$\begin{aligned}
&\vec{w}(x,0) = \vec{\phi}(x)\\
&\vec{w}(0,t) = \int_0^t \vec{z}(0,\tau)d\tau + \vec{\phi}(0) = \int_0^t \vec{h}_1'(\tau)d\tau + \vec{\phi}(0) = \vec{h}_1(t) - \vec{h}_1(0) + \vec{\phi}(0) = \vec{h}_1(t)\\
&\vec{w}(1,t) = \int_0^t \vec{z}(1,\tau)d\tau + \vec{\phi}(1) = \int_0^t \vec{h}_2'(\tau)d\tau + \vec{\phi}(1) = \vec{h}_2(t) - \vec{h}_2(0) + \vec{\phi}(1) = \vec{h}_2(t)\\
&\vec{w}_x(1,t) = \int_0^t \vec{z}_x(1,\tau)d\tau + \vec{\phi}'(1) = \int_0^t \vec{h}_3'(\tau)d\tau + \vec{\phi}'(1) = \vec{h}_3(t) - \vec{h}_3(0) + \vec{\phi}'(1) = \vec{h}_3(t)
\end{aligned}$$

then

$$\vec{w}_t(x,t)+\vec{w}_{xxx}(x,t)=\vec{z}(x,t)+\int_0^t \vec{z}_{xxx}(x,\tau)d\tau+\vec{\phi}'''(x)$$
$$=\vec{z}(x,0)+\int_0^t \vec{F}_t(x,\tau)d\tau+\vec{\phi}'''(x)=\vec{F}(x,t)$$

thus $\vec{w}(x,t)$ is a solution of system (3.5)-(3.7), then we have

$$\vec{w}_{xxx}=\vec{F}-\vec{w}_t=\vec{F}-\vec{z}$$

thus $\vec{w}\in Y_{3,T}$ and (3.8) holds with $s=3$.

The estimate(3.8) holds for $0\le s\le 3$ will be established by invoking Lemma3.6. Let $B_0^1\times B_0^1=X_{0,T}\times X_{0,T}\times L_1(0,T;L_2(0,1))\times L_1(0,T;L_2(0,1)), B_1^1\times B_1^1=X_{3,T}\times X_{3,T}\times W^{1,1}(0,T;L_2(0,1))\times W^{1,1}(0,T;L_2(0,1)), B_0^2\times B_0^2=Y_{0,T}, B_1^2\times B_1^2=Y_{3,T}$. Let A be the solution map of the IBVP(3.5)-(3.7): $\vec{\omega}=A(\vec{\phi},\vec{h}_1,\vec{h}_2,\vec{h}_3,\vec{F})$. For given $0<s<3$, choose $p=2,\theta=\frac{s}{3}$. Then, $B_{\theta,p}^2\times B_{\theta,p}^2=Y_{s,T}, B_{\theta,p}^1\times B_{\theta,p}^1=X_{s,T}\times X_{s,T}\times W^{\frac{s}{3},1}(0,T;L_2(0,1))\times W^{\frac{s}{3},1}(0,T;L_2(0,1))$. In this case, assumption(ii) of Lemma3.6 has been proved with $s=3$. So we only need to verify assumption(i). Let $\vec{\omega}_1=A(\vec{\phi}_1,\vec{h_{11}},\vec{h_{21}},\vec{h_{31}},\vec{F}_1),\vec{\omega}_2=A(\vec{\phi}_2,\vec{h_{12}},\vec{h_{22}},\vec{h_{32}},\vec{F}_2),\vec{z}=\vec{\omega}_1-\vec{\omega}_2$. Then $\vec{z}$ satisfies the following problem:

$$\vec{z}_t+\vec{z}_{xxx}=\vec{F}_1-\vec{F}_2$$
$$\vec{z}(x,0)=\vec{\phi}_1-\vec{\phi}_2$$
$$\vec{z}(0,t)=\vec{h_{11}}-\vec{h_{12}},\vec{z}(1,t)=\vec{h_{21}}-\vec{h_{22}},\vec{z}_x(1,t)=\vec{h_{31}}-\vec{h_{32}}.$$

Similar to the proof of Lemma3.1, the result for values of $0\le s\le 3$ can be established by Lemma3.6 and (3.9).

For a given $s\in R$ and Ω being R^+ or a finite interval $(0,L)$, we define $H^s(\Omega)$ as the restriction of the space $H^s(R)$ to $\Omega:$

$$H^s(\Omega)=\{f|_\Omega \mid f\in H^s(R)\}$$

with

$$\|f\|_{H^s(\Omega)}=inf\{\|\tilde{f}\|_{H^s(R)}|\tilde{f}\in H^s(R),\tilde{f}|_\Omega=f\}.$$

Other equivalent definitions of $H^s(\Omega)$ can be found in [16]. The space $H_0^s(R^+)$ is the subspace which is the closure of the class of functions in $H^s(R)$ whose support lies in R^+. It transpires that the standard interpolation space $[H_1(R^+), L^2(R^+)]_\theta = H^\theta(R^+)$ for $0 \le \theta \le 1$ and that $[H_0^1(R^+), L^2(R^+)]_\theta = H_\theta^0(R^+)$ for $0 \le \theta \le 1, \theta \ne \frac{1}{2}$. For the special case $\theta = \frac{1}{2}$, this interpolation gives the Lions-Magenes space $H_{00}^{\frac{1}{2}}$ which is neither algebraically nor topologically equivalent to $H_0^{\frac{1}{2}}(R^+)$. Similar remarks apply to bounded domains. So, for the convenience of our discussion of the traces of functions in $H^s(R)$, we cut two points for $s = \frac{1}{2}, s = \frac{3}{2}$. The proof is complete. $\square$

Lemma 3.9 *For given $T > 0, s \ge 0$. Assume that $(\vec{\phi}, \vec{h_1}, \vec{h_2}, \vec{h_3}) \in X_{s,T} \times X_{s,T}$ satisfies the s-compatibility conditions, then there exists a $T^* \in (0, T]$ such that the IBVP* (3.1)-(3.3) *admits a unique solution $\vec{w} \in Y_{s,T^*}$, where T^* relies on $\|(\vec{\phi}, \vec{h_1}, \vec{h_2}, \vec{h_3})\|_{X_{s,T} \times X_{s,T}}$, and $\partial_t^j \vec{w} \in Y_{s-3j,T^*}, j = 0, 1, 2, \ldots, [\frac{s}{3}] - 1, [\frac{s}{3}]$. Furthermore, for any $T' < T^*$, there exists a neighborhood U of $(\vec{\phi}, \vec{h_1}, \vec{h_2}, \vec{h_3})$ such that the IBVP* (3.1)-(3.3) *admits a unique solution in the space $Y_{s,T'}$, where $(\vec{\psi}, \vec{H}_1, \vec{H}_2, \vec{H}_3) \in U$ and the corresponding solution map from U to $Y_{s,T'}$ is Lipschitz continuous.*

Proof. Let $(\vec{\phi}, \vec{h_1}, \vec{h_2}, \vec{h_3}) \in X_{s,T} \times X_{s,T}$ satisfy the s-compatibility conditions, $r > 0, \theta > 0. S_{\theta,r}$ is a collection of functions $\vec{z}$ in the space $(C([0,\theta]; L_2(0,1)) \cap L_2(0,\theta; H^1(0,1))) \times (C([0,\theta]; L_2(0,1)) \cap L_2(0,\theta; H^1(0,1)))$ satisfying

$$\partial_t^j \vec{z} \in Y_{3,\theta}, j = 0, 1, 2, \ldots, [\frac{s}{3}] - 1.$$

$$\partial_t^{[\frac{s}{3}]} \vec{z} \in Y_{s-3[\frac{s}{3}],\theta},$$

and

$$\|\partial_t^{[\frac{s}{3}]} \vec{z}\|_{Y_{s-3[\frac{s}{3}],\theta}} + \sum_{j=0}^{[\frac{s}{3}]-1} \|\partial_t^j \vec{z}\|_{Y_{3,\theta}} \le r$$

define

$$\mathcal{Y}_{s,\theta} = Y_{s-3[\frac{s}{3}],\theta} \times \Pi_{j=0}^{[\frac{s}{3}]-1} Y_{3,\theta}$$

then the set $S_{\theta,r}$ is a closed, convex and bounded subset of the space $\mathcal{Y}_{s,\theta}$ via the mapping $\vec{z} \to (\vec{z}, \partial_t \vec{z}, \ldots, \partial_t^{[\frac{s}{3}]} \vec{z}) \equiv Z$ and therefore is a complete metric space in the topology induced from $\mathcal{Y}_{s,\theta}$. For any given $\vec{z} \in S_{\theta,r}$, consider the following system

$$\vec{w}_t^{(k)} + \vec{w}_{xxx}^{(k)} = \vec{E}(x,t) \tag{3.13}$$

$$\vec{w}(x,0)^{(k)} = \vec{\phi}_k(x) \tag{3.14}$$

$$\vec{w}^{(k)}(0,t) = \vec{h}_1^{(k)}(t), \vec{w}^{(k)}(1,t) = \vec{h}_2^{(k)}(t), \vec{w}_x^{(k)}(1,t) = \vec{h}_3^{(k)}(t) \tag{3.15}$$

where

$$\begin{aligned} &\vec{E}(x,t) \\ &= -(3\partial_x(\sum_{j=0}^{k} \frac{k!}{j!(k-j)!} z_1^{(j)} z_1^{(k-j)}) - b\partial_x(\sum_{j=0}^{k} \frac{k!}{j!(k-j)!} z_2^{(j)} z_2^{(k-j)}), 3\partial_x(\sum_{j=0}^{k} \frac{k!}{j!(k-j)!} z_1^{(j)} \partial_x z_2^{(k-j)})^T \end{aligned} \tag{3.16}$$

$k = 0, 1, 2, \ldots, [\frac{s}{3}]$ and $\vec{w}^{(k)} \equiv \partial_t^k \vec{w}, z_1^{(k)} \equiv \partial_t^k z_1, z_2^{(k)} \equiv \partial_t^k z_2, \vec{h}_j^{(k)}(t)$ is the k-th order derivative of $\vec{h}_j (j = 1, 2, 3)$ and

$$\vec{\phi}_0(x) = \vec{\phi}(x)$$

$$\vec{\phi}_k(x) = -(\vec{\phi}'''_{k-1}(x) + \vec{\phi}'_{k-1}(x) + \sum_{j=0}^{k-1} (\vec{\phi}_j(x)\vec{\phi}_{k-j-1}(x))')$$

satisfy the following compatibility conditions

$$\vec{\phi}_k(0) = \vec{h}_1^{(k)}(0), \vec{\phi}_k(1) = \vec{h}_2^{(k)}(0), \vec{\phi}'_k(1) = \vec{h}_3^{(k)}(0)$$

define a map Γ from $S_{\theta,r}$ to the space $\mathcal{Y}_{s,\theta}$, similar to Lemma 3.3, we have

$$\|\Gamma(\vec{z})\|_{\mathcal{Y}_{s,\theta}} \leq C\|(\vec{\phi}, \vec{h_1}, \vec{h_2}, \vec{h_3})\|_{X_{s,T} \times X_{s,T}} + C(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}})\|\vec{z}\|_{\mathcal{Y}_{s,\theta}}$$

where $\vec{z} = (z_1, z_2)^T, C$ is a constant independent of $\vec{h_1}, \vec{h_2}, \vec{h_3}, \vec{\phi}$ and θ. Thus, with an appropriate choice of r and θ, the map Γ is a contraction mapping of $S_{\theta,r}$. Its fixed point $\vec{w} = \Gamma(\vec{w}) \in S_{\theta,r}$ is the unique solution of (3.5)-(3.7). Thus in case $s \leq 3$, the proof is complete.

When $s > 3$, we have:

$$\partial_t^{(j)} \vec{w} \in (C([0,\theta]; H^3(0,1)) \cap L_2([0,\theta]; H^4(0,1))) \times (C([0,\theta]; H^3(0,1)) \cap L_2([0,\theta]; H^4(0,1)))$$

$$\partial_t^{[\frac{s}{3}]}\vec{w}\in Y_{s-3[\frac{s}{3}],\theta}\subset \begin{aligned}[t] &(C([0,\theta];H^{s-3[\frac{s}{3}]}(0,1))\cap L_2([0,\theta];H^{s+1-3[\frac{s}{3}]}(0,1)))\\ &\times(C([0,\theta];H^{s-3[\frac{s}{3}]}(0,1))\cap L_2([0,\theta];H^{s+1-3[\frac{s}{3}]}(0,1)))\end{aligned}$$

where $j=0,1,\ldots,[\frac{s}{3}]-1$ and $k=[\frac{s}{3}]-1$, we get

$$\begin{aligned}\vec{w}_{xxx}^{([\frac{s}{3}]-1)} &= -\vec{w}_t^{([\frac{s}{3}]-1)}-(3\partial_x(\sum_{j=0}^{([\frac{s}{3}]-1)}C^j_{([\frac{s}{3}]-1)}z_1^{(j)}z_1^{([\frac{s}{3}]-1-j)})\\ &-b\partial_x(\sum_{j=0}^{([\frac{s}{3}]-1)}C^j_{([\frac{s}{3}]-1)}z_2^{(j)}z_2^{([\frac{s}{3}]-1-j)}),3\partial_x(\sum_{j=0}^{k}C^j_{([\frac{s}{3}]-1)}z_1^{(j)}\partial_x z_2^{([\frac{s}{3}]-1-j)})^T\end{aligned}$$

then we have

$$\vec{w}^{([\frac{s}{3}]-1)}\in \begin{aligned}[t] &(C([0,\theta];H^{s-3[\frac{s}{3}]+3}(0,1))\cap L_2([0,\theta];H^{s+4-3[\frac{s}{3}]}(0,1)))\\ &\times(C([0,\theta];H^{s-3[\frac{s}{3}]+3}(0,1))\cap L_2([0,\theta];H^{s+4-3[\frac{s}{3}]}(0,1)))\end{aligned}$$

which means

$$\begin{aligned}\vec{w}\in\ &(C([0,\theta];H^s(0,1))\cap L_2([0,\theta];H^{s+1}(0,1)))\times(C([0,\theta];H^s(0,1))\cap L_2([0,\theta];H^{s+1}(0,1)))\\ \partial_t^j\vec{w}\in\ &(C([0,\theta];H^{s-3j}(0,1))\cap L_2([0,\theta];H^{s+1-3j}(0,1)))\times(C([0,\theta];H^{s-3j}(0,1))\\ &\cap L_2([0,\theta];H^{s+1-3j}(0,1)))\end{aligned}$$

where $j=1,2,\ldots,[\frac{s}{3}]-1$ and

$$\partial_t^{[\frac{s}{3}]}\vec{w}\in \begin{aligned}[t] &(C([0,\theta];H^{s-3[\frac{s}{3}]}(0,1))\cap L_2([0,\theta];H^{s+1-3[\frac{s}{3}]}(0,1)))\\ &\times(C([0,\theta];H^{s-3[\frac{s}{3}]}(0,1))\cap L_2([0,\theta];H^{s+1-3[\frac{s}{3}]}(0,1)))\end{aligned}$$

The proof is complete. □

4 Global well-posedness

In this section, we discuss the global well-posedness of the nonlinear IBVP (3.1)-(3.3). For any fixed $T>0$, $s\geq 0$, and $\epsilon>0$, we set

$$Z_{s,T}\equiv H^s(0,1)\times H^{\epsilon+\frac{5s+9}{18}}(0,T)\times H^{\epsilon+\frac{5s+9}{18}}(0,T)\times H^{\epsilon+\frac{5s+3}{18}}(0,T),\quad (0\leq s\leq 3);$$

$$Z_{s,T}\equiv X_{s,T},\quad (s\geq 3).$$

Then, we have the following prior H^s-estimate for global smooth solutions of the nonlinear IBVP(3.1)-(3.3).

Lemma 4.1 *Assume $s \geq 0, T > 0$, then for any smooth solution to the nonlinear IBVP(3.1)-(3.3), denoted $\vec{\omega}$, we have:*

$$sup_{0 \leq t \leq T} \|\vec{\omega}(\cdot, t)\|_{H^s(0,1)\times H^s(0,1)} \leq C\|(\vec{\phi}, \vec{h_1}, \vec{h_2}, \vec{h_3})\|_{Z_{s,T}\times Z_{s,T}}, \tag{4.1}$$

where C is a constant.

Proof. First, we consider the case that $s = 0$. We divide $\vec{\omega}$ into two parts as $\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$ such that $\vec{\omega}_1$ satisfies:

$$\vec{\omega}_{1t} + \vec{\omega}_{1xxx} = \vec{0}, \tag{4.2}$$

$$\vec{\omega}_1(x, 0) = \vec{\psi}(x), \tag{4.3}$$

$$\vec{\omega}_1(0, t) = \vec{h_1}(t), \vec{\omega}_1(1, t) = \vec{h_2}(t), \vec{\omega}_{1x}(1, t) = \vec{h_3}(t), \tag{4.4}$$

where

$$\vec{\psi}(x) = (1 - x)\vec{h_1}(0) + x\vec{h_2}(0) + x(1 - x)(\vec{h_3}(0) - \vec{h_2}(0) + \vec{h_1}(0)), \vec{\omega}_1 = (u_1, v_1)^T;$$

and $\vec{\omega}_2$ satisfies

$$\vec{\omega}_{2t} + \vec{\omega}_{2xxx} = -B(\vec{\omega}_1 + \vec{\omega}_2), \tag{4.5}$$

$$\vec{\omega}_2(x, 0) = \vec{\phi}(x) - \vec{\psi}(x), \tag{4.6}$$

$$\vec{\omega}_2(0, t) = \vec{0}, \vec{\omega}_2(1, t) = \vec{0}, \vec{\omega}_{2x}(1, t) = \vec{0}, \tag{4.7}$$

where $B = \begin{pmatrix} 6u_x & -2bv_x \\ 3v_x & 0 \end{pmatrix}, \vec{\omega}_2 = (u_2, v_2)^T$, then $u = u_1 + u_2, v = v_1 + v_2$. According to Lemma 3.8, we see

$$\|\vec{\omega}_1\|_{Y_{s,T}} \leq C\|(\vec{\psi}, \vec{h_1}, \vec{h_2}, \vec{h_3})\|_{X_{s,T}\times X_{s,T}}.$$

By Multiplying (4.5) by $\vec{\omega}_2$ and then integrating the product over $(0, 1)$ with respect to x, we have

$$\frac{d}{dt}\|\vec{\omega}_2(\cdot, t)\|^2_{L_2(0,1)\times L_2(0,1)} \leq \ \|B\vec{\omega}_2\|_{L_2(0,1)\times L_2(0,1)}\|\vec{\omega}_1\|_{L_2(0,1)\times L_2(0,1)}$$

$$+\|B\vec{\omega}_2\|_{L_2(0,1)\times L_2(0,1)}\|\vec{\omega}_2\|_{L_2(0,1)\times L_2(0,1)},$$

with a similar discussing in the proof of Lemma 3.1, the following can be deduced:

$$\|B\vec{\omega}_2\|_{L_2(0,1)\times L_2(0,1)} \leq C\|\vec{\omega}_2\|_{L_2(0,1)\times L_2(0,1)}.$$

Then, by applying the Gronwall's inequality, the estimate (4.1) with $s = 0$ follows.

Next, for the cases that $s = 3$. Let $\vec{z} = \vec{\omega}_t$, such that $\vec{z}$ satisfies:

$$\vec{z}_t + \vec{z}_{xxx} + B_t\vec{\omega} + B\vec{z} = \vec{0}, \tag{4.8}$$

$$\vec{z}(x,0) = \vec{\phi}^*, \tag{4.9}$$

$$\vec{z}(0,t) = \vec{h_1}'(t), \vec{z}(1,t) = \vec{h_2}'(t), \vec{z}_x(1,t) = \vec{h_3}'(t), \tag{4.10}$$

where $\vec{\omega} = (u,v)^T, \vec{z} = \vec{\omega}_t = (u_t,v_t)^T, B = \begin{pmatrix} 6u_x & -2bv_x \\ 3v_x & 0 \end{pmatrix}, B_t = \begin{pmatrix} 6u_{tx} & -2bv_{tx} \\ 3v_{tx} & 0 \end{pmatrix}, \vec{\phi}^*(x) = -\vec{\phi}'''(x) - B\vec{\phi}(x)$. Then, for the system(4.8)-(4.10), there exists a constant $C > 0$ such that for any $T' \leq T$,

$$\|\vec{z}\|_{Y_{0,T'}} \leq C\|(\vec{\phi}^*, \vec{h}_1', \vec{h}_2', \vec{h}_3')\|_{X_{0,T}\times X_{0,T}} + C(T'^{\frac{1}{2}} + T'^{\frac{1}{3}})\|\vec{\omega}\|_{Y_{0,T}}\|\vec{z}\|_{Y_{0,T}}.$$

Let $T' < T$ satisfies $C(T'^{\frac{1}{2}} + T'^{\frac{1}{3}})\|\vec{\omega}\|_{Y_{0,T}} = \frac{1}{2}$, then

$$\|\vec{z}\|_{Y_{0,T'}} \leq 2C\|(\vec{\phi}^*, \vec{h}_1', \vec{h}_2', \vec{h}_3')\|_{X_{0,T}\times X_{0,T}}.$$

Since T' depends on $\|\vec{\omega}\|_{Y_{0,T}}$, it depends on $\|(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3)\|_{Z_{0,T}\times Z_{0,T}}$. Therefore, we have

$$\|\vec{z}\|_{Y_{0,T}} \leq C_1\|(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3)\|_{Z_{3,T}\times Z_{3,T}},$$

where C_1 is a constant depends on T and $\|(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3)\|_{Z_{0,T}\times Z_{0,T}}$. Since $\vec{z} = -(\vec{\omega}_{xxx} + B\vec{\omega})$, the estimate (4.1) with $s = 3$ follows.

The estimate (4.1) for values of $0 < s < 3$ can be established by Lemma 3.6, which is a consequence in interpolation theory. To see this, we take $B_0^1 \times B_0^1 = Z_{0,T} \times Z_{0,T}, B_1^1 \times B_1^1 = Z_{3,T} \times Z_{3,T}, B_0^2 \times B_0^2 = C([0,T]; L_2(0,1)) \times C([0,T]; L_2(0,1)), B_1^2 \times B_1^2 = C([0,T]; H^3(0,1)) \times C([0,T]; H^3(0,1))$. We assume $(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3)$ satisfy s-compatibility condition.

Let A be the solution map of the IBVP(3.1)-(3.3): $\vec{\omega} = A(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3)$. For given s with $0 < s < 3$, choose $p = 2, \theta = \frac{s}{3}$. Then $B_{\theta,p}^2 \times B_{\theta,p}^2 = C([0,T]; H^s(0,1)) \times C([0,T]; H^s(0,1)), B_{\theta,p}^2 \times$

$B^2_{\theta,p} = Z_{s,T} \times Z_{s,T}$. In this case, assumption(ii) of Lemma 3.6 corresponds to (4.1) with $s = 3$, which we have already proved. So we only need to verify assumption(i).

Let $\vec{\omega}_1 = A(\vec{\phi}_1, \vec{h}_{11}, \vec{h}_{21}, \vec{h}_{31}), \vec{\omega}_2 = A(\vec{\phi}_2, \vec{h}_{12}, \vec{h}_{22}, \vec{h}_{32}), \vec{\xi} = \vec{\omega}_1 - \vec{\omega}_1, \vec{\omega}_1 = (u_1, v_1)^T, \vec{\omega}_2 = (u_2, v_2)^T, B_1 = \begin{pmatrix} 6u_{1x} & -2bv_{1x} \\ 3v_{1x} & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 6u_{2x} & -2bv_{2x} \\ 3v_{2x} & 0 \end{pmatrix}$ satisfy

$$\begin{aligned}
&\vec{\xi}_t + \vec{\xi}_{xxx} + (B_1\vec{\omega}_1 - B_2\vec{\omega}_2) = \vec{0} \\
&\vec{\xi}(x,0) = \vec{\phi}_1(x) - \vec{\phi}_2(x) \\
&\vec{\xi}(0,t) = \vec{h}_{11}(t) - \vec{h}_{12}(t), \vec{\xi}(1,t) = \vec{h}_{21}(t) - \vec{h}_{22}(t), \vec{\xi}(1,t)_x = \vec{h}_{31}(t) - \vec{h}_{32}(t).
\end{aligned}$$

By Lemma 3.8, we see for any $0 \le T' \le T$,

$$\begin{aligned}
\|\vec{\xi}\|_{Y_{0,T'}} \le\ & C(\|(\vec{\phi}_1, \vec{h}_{11}, \vec{h}_{21}, \vec{h}_{31}) - (\vec{\phi}_2, \vec{h}_{12}, \vec{h}_{22}, \vec{h}_{32})\|_{X_{0,T}\times X_{0,T}} \\
& + \|B_1\vec{\omega}_1 - B_2\vec{\omega}_2\|_{L_1(0,T';L_2(0,1))\times L_1(0,T';L_2(0,1))}) \\
\le\ & C(\|(\vec{\phi}_1, \vec{h}_{11}, \vec{h}_{21}, \vec{h}_{31}) - (\vec{\phi}_2, \vec{h}_{12}, \vec{h}_{22}, \vec{h}_{32})\|_{X_{0,T}\times X_{0,T}} + C(T'^{\frac{1}{2}} + T'^{\frac{1}{3}})\|\vec{\xi}\|_{Y_{0,T'}}\|\vec{\xi}\|_{Y_{0,T}}.
\end{aligned}$$

With a similar discuss in the proof of $s = 3$, we can deduce:

$$\|\vec{\xi}\|_{Y_{0,T}} \le C(\|(\vec{\phi}_1, \vec{h}_{11}, \vec{h}_{21}, \vec{h}_{31})\|_{Z_{0,T}\times Z_{0,T}} + \|(\vec{\phi}_2, \vec{h}_{12}, \vec{h}_{22}, \vec{h}_{32})\|_{Z_{0,T}\times Z_{0,T}}).$$

We take T' such that $C(T'^{\frac{1}{2}} + T'^{\frac{1}{3}})\|\vec{\xi}\|_{Y_{0,T}} = \frac{1}{2}$, then we have

$$\|\vec{\xi}\|_{Y_{0,T'}} \le 2C(\|(\vec{\phi}_1, \vec{h}_{11}, \vec{h}_{21}, \vec{h}_{31}) - (\vec{\phi}_2, \vec{h}_{12}, \vec{h}_{22}, \vec{h}_{32})\|_{X_{0,T}\times X_{0,T}}).$$

Since T' only depends on $\|\vec{\xi}\|_{Y_{0,T}}$, which in turn only depends on $\|(\vec{\phi}_1, \vec{h}_{11}, \vec{h}_{21}, \vec{h}_{31})\|_{Z_{0,T}\times Z_{0,T}} + \|(\vec{\phi}_2, \vec{h}_{12}, \vec{h}_{22}, \vec{h}_{32})\|_{Z_{0,T}\times Z_{0,T}}$, one can arrive at

$$\begin{aligned}
\|\vec{\xi}\|_{Y_{0,T}} \le\ & C(\|(\vec{\phi}_1, \vec{h}_{11}, \vec{h}_{21}, \vec{h}_{31})\|_{Z_{0,T}\times Z_{0,T}} + \|(\vec{\phi}_2, \vec{h}_{12}, \vec{h}_{22}, \vec{h}_{32})\|_{Z_{0,T}\times Z_{0,T}}) \cdot \\
& \|(\vec{\phi}_1, \vec{h}_{11}, \vec{h}_{21}, \vec{h}_{31}) - (\vec{\phi}_2, \vec{h}_{12}, \vec{h}_{22}, \vec{h}_{32})\|_{X_{0,T}\times X_{0,T}}.
\end{aligned}$$

by a standard extension argument. Therefore, we see assumption(i) of Lemma 3.6 is satisfied. Estimate (4.1) for $0 < s < 3$ is established by invoking Lemma 3.1.

Next, we give the proof for the cases that $3 < s < 6$, and the cases that $s \ge 6$ can be established by the same method. Let $\vec{\omega}$ be a smooth solution of the IBVP(3.1)-(3.3), we set

$\vec{z} = \vec{\omega}_t$ satisfies(4.8)-(4.10), then for the system(4.8)-(4.10), there exists a constant $C > 0$ such that for any $0 < T' < T$,

$$\|\vec{z}\|_{Y_{s-3,T'}} \leq C\|(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3)\|_{Z_{s,T}\times Z_{s,T}} + C(T'^{\frac{1}{2}} + T'^{\frac{1}{3}})\|\vec{\omega}\|_{Y_{s-3,T}}\|\vec{z}\|_{Y_{s-3,T'}},$$

Let $T' < T$ satisfies $C(T'^{\frac{1}{2}} + T'^{\frac{1}{3}})\|\vec{\omega}\|_{Y_{s-3,T}} = \frac{1}{2}$, then

$$\|\vec{z}\|_{Y_{s-3,T'}} \leq 2C\|(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3)\|_{Z_{s,T}\times Z_{s,T}}.$$

Since T' depends on $\|vec\omega\|_{Y_{s-3,T}}$, it depends on $\|(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3)\|_{Z_{s,T}\times Z_{s,T}}$. Therefore, we have

$$\|\vec{z}\|_{Y_{s-3,T}} \leq C_1\|(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3)\|_{Z_{s,T}\times Z_{s,T}}.$$

Consequently, we have

$$\|\vec{\omega}\|_{Y_{s,T}} \leq C\|(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3)\|_{Z_{s,T}\times Z_{s,T}}.$$

This complete the proof. □

Directly, we obtain the following theorem:

Theorem 4.2 *For any $T > 0, s \geq 0$ the IBVP(3.1)-(3.3) with s-compatible condition $(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3) \in Z_{s,T} \times Z_{s,T}$, admits a unique solution $\vec{\omega} \in Y_{s,T}$ such that $\partial_t^j \vec{\omega} \in Y_{s-3j,T}$ for $j = 0, 1, \ldots, [\frac{s}{3}]$. Furthermore, the corresponding solution map is Lipschitz continuous.*

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