# ON NEW GENERALIZED QUANTUM INTEGRALS AND RELATED HERMITE-HADAMARD INEQUALITIES

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Abstract. In this article, we introduce a new concept of quantum integrals which is called ${}^{\kappa_2}T_q$-integral. Then we prove several properties of this concept of quantum integrals. Moreover, we present several Hermite-Hadamard type inequalities for ${}^{\kappa_2}T_q$-integral by utilizing differentiable convex functions. The results presented in this article are unification and generalization of the comparable results in the literature.

## 1. Introduction

In mathematics, the quantum calculus is equivalent to usual infinitesimal calculus without the concept of limits or the investigation of calculus without limits (quantum is from the Latin word "quantus" and literally it means how much, in Swedish "Kvant"). It has two major branches, $q$-calculus and the $h$-calculus. And both of them were worked out by P. Cheung and V. Kac [13] in the early twentieth century. In the same era FH Jackson started worked on Quantum Calculus or $q$-calculus, but Euler and Jacobi had already figured out this type of calculus. A number of studies have recently been widely used in the field of $q$-analysis, beginning with Euler, due to the vast necessity for mathematics that models of quantum computing $q$-calculus exist in the framework between physics and mathematics. Tariboon and Ntouyas [22] proposed the quantum calculus concepts on finite intervals and obtained several $q$-analogues of classical mathematical objects. This inspired other researchers and, as a consequence, numerous novel results concerning quantum analogues of classical mathematical results have already been launched in the literature. Noor et al. [17] obtained new $q$-analogues of inequality utilizing first order $q$-differentiable convex function. In [1], Alp et al. acquired some bonds for left hand side of $q$-Hermite Hadamard inequalities and quantum calculations by using convex and quasi-convex functions for midpoint form inequalities. For more details, see [13]-[21] and references cited therein.
In various mathematical fields, it has many applications, like number theory, combinatorics, orthogonal polynomials, simple hyper-geometric functions and other sciences, quantum theory, physics and relativity theory. Many of the fundamental aspects of quantum calculus. It has been shown that quantum calculus is a subfield of the more general mathematical field of time scales calculus. New developments have recently been made in the research and methodology of dynamic derivatives on time scales. The research offers a consolidation and application of traditional differential and difference equations. Moreover, it is a unification of the discrete theory with the continuous theory, from the a theoretical perspective. Time scales provide a unified framework for studying dynamic equations on both discrete and continuous domains. In studying quantum calculus, we are concerned with a specific time scale, called the $q$-time scale, defined as follows: $T := q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0\}$ see [1]-[10] and references cited therein. The Hermite-Hadamard inequality was introduced by Hermite and Hadamard, see [11]. It's one of the most recognized inequalities in the theory of convex functional analysis, which is stated as follows:

Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex mapping and $\kappa_1 < \kappa_2$. Then

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,d\varkappa \leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \tag{1.1}$$

If $\mathcal{F}$ is concave, both inequalities hold in the reverse direction.

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The important purpose of this article is to derive some new quantum integral Inequalities of the convex function for midpoint formula. Moreover, when $q \to 1$, several examples of Hermite-Hadamard form inequalities are derived as special cases.

## 2. Preliminaries of $q$-Calculus and Some Inequalities

Several fundamental inequalities that are well known in classical analysis, like Hölder inequality, Ostrowski inequality, Cauchy-Schwarz inequality, Grüss-Chebyshev inequality, Grüss inequality. Using classical convexity, other basic inequalities have been proven and applied to $q$-calculus. For more details, please see, [1, 3, 7, 9, 16, 17, 20, 23].

In this section, we discuss some required definitions of quantum calculus and important quantum integral inequalities for Hermite-Hadamard on left and right sides bonds.

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}, \quad q \in (0,1).$$

Jackson derived the $q$-Jackson integral in [12] from 0 to $\kappa_2$ for $0 < q < 1$ as follows::

$$\int_0^{\kappa_2} \mathcal{F}(\varkappa) \ d_q\varkappa \ = (1-q)\,\kappa_2 \sum_{n=0}^{\infty} q^n \mathcal{F}(\kappa_2 q^n) \tag{2.1}$$

provided the sum converge absolutely.
The $q$-Jackson integral in a generic interval $[\kappa_1, \kappa_2]$ was given by in [12] and defined as follows

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \ d_q\varkappa \ = \int_0^{\kappa_2} \mathcal{F}(\varkappa) \ d_q\varkappa \ - \int_0^{\kappa_1} \mathcal{F}(\varkappa) \ d_q\varkappa \ .$$

**Definition 1.** [22] *We suppose that a function $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is continuous. Then $q_{\kappa_1}$-derivative of $\mathcal{F}$ at $\varkappa \in [\kappa_1, \kappa_2]$ is defined as follows*

$$ {}_{\kappa_1}D_q\mathcal{F}(\varkappa) \ = \frac{\mathcal{F}(\varkappa) - \mathcal{F}(q\varkappa + (1-q)\,\kappa_1)}{(1-q)\,(\varkappa - \kappa_1)}, \ \varkappa \neq \kappa_1. \tag{2.2}$$

*Since $\mathcal{F}$ is a continuous function from $[\kappa_1, \kappa_2]$ to $\mathbb{R}$, so ${}_{\kappa_1}D_q\mathcal{F}(\kappa_1) = \lim_{\varkappa \to \kappa_1} {}_{\kappa_1}D_q\mathcal{F}(\varkappa)$ . The function $\mathcal{F}$ is said to be $q$- differentiable on $[\kappa_1, \kappa_2]$ if ${}_{\kappa_1}D_q\mathcal{F}(t)$ exists for all $\varkappa \in [\kappa_1, \kappa_2]$. If $\kappa_1 = 0$ in (2.2), then ${}_0D_q\mathcal{F}(\varkappa) = D_q\mathcal{F}(\varkappa)$ , where $D_q\mathcal{F}(\varkappa)$ is familiar $q$-derivative of $\mathcal{F}$ at $\varkappa \in [\kappa_1, \kappa_2]$ defined by the expression (see [13])*

$$D_q\mathcal{F}(\varkappa) \ = \frac{\mathcal{F}(\varkappa) - \mathcal{F}(q\varkappa)}{(1-q)\,\varkappa}, \ \varkappa \neq 0.$$

**Definition 2.** [4] *We suppose that a function $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is continuous, then $q^{\kappa_2}$-derivative of $\mathcal{F}$ at $\varkappa \in [\kappa_1, \kappa_2]$ is defined as follows*

$$ {}^{\kappa_2}D_q\mathcal{F}(\varkappa) \ = \frac{\mathcal{F}(q\varkappa + (1-q)\,\kappa_2) - \mathcal{F}(\varkappa)}{(1-q)\,(\kappa_2 - \varkappa)}, \ \varkappa \neq \kappa_2.$$

**Definition 3.** [22] *We suppose that a function $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is continuous, then the $q_{\kappa_1}$-definite integral on $[\kappa_1, \kappa_2]$ is defined as follows*

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \ {}_{\kappa_1}d_q\varkappa \ = (1-q)\,(\kappa_2 - \kappa_1) \sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\kappa_2 + (1-q^n)\,\kappa_1) = (\kappa_2 - \kappa_1) \int_0^1 \mathcal{F}((1-t)\,\kappa_1 + t\kappa_2) \ d_qt \ .$$

In [1], Alp et al. established the $q_{\kappa_1}$-Hermite-Hadamard inequalities for convexity, which is defined as follows,

**Theorem 1.** *Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex differentiable function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then $q$-Hermite-Hadamard inequalities are as follows:*

$$\mathcal{F}\left(\frac{q\kappa_1 + \kappa_2}{1+q}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \ {}_{\kappa_1}d_q\varkappa \ \leq \frac{q\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{1+q}. \tag{2.3}$$

The authors of [15] and [1] have set certain boundaries for the left and right sides of the inequality (2.3).
On the other hand, the following new description and related Hermite-Hadamard form inequalities were given by Bermudo et al.:

**Definition 4.** [4] *Let $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a continuous function. Then, the $q^{\kappa_2}$-definite integral on $[\kappa_1,\kappa_2]$ is defined as*

$$\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\ {}_{\kappa_2}d_q\varkappa = (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_1+(1-q^n)\kappa_2)=(\kappa_2-\kappa_1)\int_0^1\mathcal{F}(t\kappa_1+(1-t)\kappa_2)\ d_qt\ .$$

**Theorem 2.** [4] *Let $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a convex function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then, q-Hermite-Hadamard inequalities are as follows:*

$$\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{1+q}\right)\le\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\ {}_{\kappa_2}d_q\varkappa\le\frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{1+q}. \tag{2.4}$$

From Theorem 1 and Theorem 2, one can the following inequalities:

**Corollary 1.** [4] *For any convex function $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ and $0<q<1$, we have*
(2.5)

$$\mathcal{F}\left(\frac{q\kappa_1+\kappa_2}{1+q}\right)+\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{1+q}\right)\le\frac{1}{\kappa_2-\kappa_1}\left\{\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\ {}_{\kappa_1}d_q\varkappa+\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\ {}_{\kappa_2}d_q\varkappa\right\}\le\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)$$

*and*

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\le\frac{1}{2(\kappa_2-\kappa_1)}\left\{\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\ {}_{\kappa_1}d_q\varkappa+\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\ {}_{\kappa_2}d_q\varkappa\right\}\le\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \tag{2.6}$$

Alp and Sarikaya, by using the area of trapezoids, introduced the following generalized quantum integral which we will called ${}_{\kappa_1}T_q$-integral.

**Definition 5.** [2] *Let $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ is continuous function. For $\varkappa\in[\kappa_1,\kappa_2]$*

$$\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\xi)\ {}_{\kappa_1}d_q^T\xi=\frac{(1-q)(\kappa_2-\kappa_1)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_2+(1-q^n)\kappa_1)-\mathcal{F}(\kappa_2)\right] \tag{2.7}$$

*where $0<q<1$.*

**Theorem 3** ($\overline{q}$-Hermite-Hadamard)**.** [2] *Let $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a convex continuous function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then we have*

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\le\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\ {}_{\kappa_1}d_q^T\varkappa\le\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2} \tag{2.8}$$

## 3. New generilazed Quantum Integrals

In this section, we introduce to a new generalized quantum integral which is called ${}_{\kappa_2}T_q$-integral. We also prove several properties of this integral.

As can see from Figure 1, the area of $n^{th}$-trapezoid is

$$B_n=(1-q)q^n(\kappa_2-\kappa_1)\frac{\mathcal{F}\left(q^{n+1}\kappa_1+\left(1-q^{n+1}\right)\kappa_2\right)+\mathcal{F}(q^n\kappa_1+(1-q^n)\kappa_2)}{2}.$$

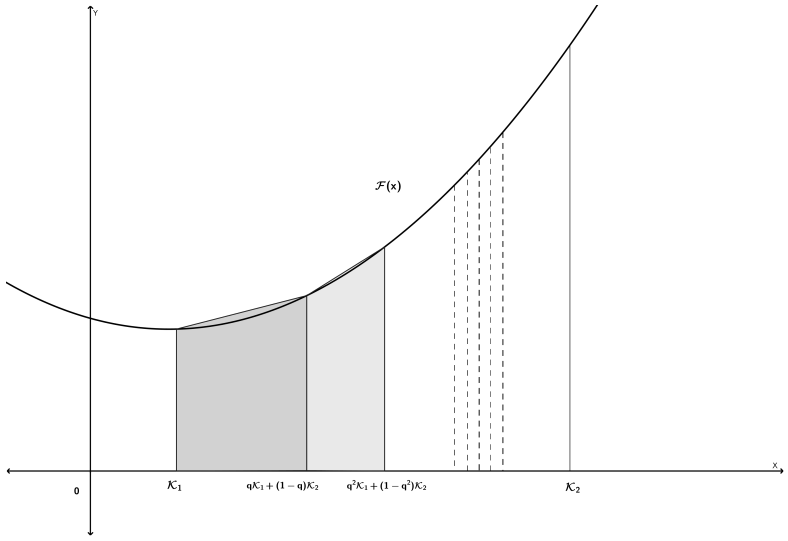


FIGURE 1. Areas of Trapezoids

By summing of the all area of $B_n$, $n = 1, 2, ...$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n &= \frac{(1-q)(\kappa_2-\kappa_1)}{2}\left[\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^{n+1}\kappa_1+\left(1-q^{n+1}\right)\kappa_2\right)+\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)\right] \\
&= \frac{(1-q)(\kappa_2-\kappa_1)}{2}\left[\frac{1}{q}\sum_{n=0}^{\infty} q^{n+1} \mathcal{F}\left(q^{n+1}\kappa_1+\left(1-q^{n+1}\right)\kappa_2\right)+\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)\right] \\
&= \frac{(1-q)(\kappa_2-\kappa_1)}{2}\left[\frac{1}{q}\sum_{n=1}^{\infty} q^n \mathcal{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)+\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)\right] \\
&= \frac{(1-q)(\kappa_2-\kappa_1)}{2}\left[\frac{1}{q}\left\{\mathcal{F}(\kappa_1)-\mathcal{F}(\kappa_1)+\sum_{n=1}^{\infty} q^n \mathcal{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)\right\}+\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)\right] \\
&= \frac{(1-q)(\kappa_2-\kappa_1)}{2}\left[\frac{1}{q}\left\{-\mathcal{F}(\kappa_1)+\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)\right\}+\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)\right] \\
&= \frac{(1-q)(\kappa_2-\kappa_1)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)-\mathcal{F}(\kappa_1)\right] \\
&= \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\xi)\ {}^{\kappa_2}d_q^T\xi\ .
\end{aligned}$$

Now we can give the following definition.

**Definition 6.** *Let $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ is continuous function. For $\varkappa\in[\kappa_1,\kappa_2]$,*

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\xi)\ {}^{\kappa_2}d_q^T\xi\ = \frac{(1-q)(\kappa_2-\kappa_1)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)-\mathcal{F}(\kappa_1)\right] \tag{3.1}$$

*where $0<q<1$. This integral is called ${}^{\kappa_2}T_q$-integral.*

**Theorem 4.** *Let $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a continuous function. Then we have*

$${}^{\kappa_2}D_q \int_{\varkappa}^{\kappa_2} \mathcal{F}(\xi)\ {}^{\kappa_2}d_q^T\xi\ = -\frac{\mathcal{F}(\varkappa)+\mathcal{F}(q\varkappa+(1-q)\kappa_2)}{2} \tag{3.2}$$

*for* $\varkappa \in [\kappa_1, \kappa_2]$.

*Proof.* From definition of ${}^{\kappa_2}T_q$-integral, we have

$$\int_{\varkappa}^{\kappa_2} \mathcal{F}(\xi) \ {}^{\kappa_2}d_q^T\xi = \frac{(1-q)(\kappa_2-\varkappa)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\varkappa + (1-q^n)\kappa_2\right) - \mathcal{F}(\varkappa)\right].$$

From the Definition 2, we obtain

$$\begin{aligned}
& {}^{\kappa_2}D_q \int_{\varkappa}^{\kappa_2} \mathcal{F}(\xi) \ {}^{\kappa_2}d_q^T\xi \\
= \ & {}^{\kappa_2}D_q \left\{\frac{(1-q)(\kappa_2-\varkappa)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\varkappa + (1-q^n)\kappa_2\right) - \mathcal{F}(\varkappa)\right]\right\} \\
= \ & \frac{1}{(1-q)(\kappa_2-\varkappa)}\left\{\frac{(1-q)(\kappa_2-\varkappa)q}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^{n+1}\varkappa + \left(1-q^{n+1}\right)\kappa_2\right) - \mathcal{F}\left(q\varkappa + (1-q)\kappa_2\right)\right]\right. \\
& \left. -\frac{(1-q)(\kappa_2-\varkappa)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\varkappa + (1-q^n)\kappa_2\right) - \mathcal{F}(\varkappa)\right]\right\} \\
= \ & \frac{1}{2q}\left[(1+q)\left(q\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^{n+1}\varkappa + \left(1-q^{n+1}\right)\kappa_2\right) - \sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\varkappa + (1-q^n)\kappa_2\right)\right)\right. \\
& \left. \mathcal{F}(\varkappa) - q\mathcal{F}\left(q\varkappa + (1-q)\kappa_2\right)\right] \\
= \ & -\frac{\mathcal{F}(\varkappa) + \mathcal{F}\left(q\varkappa + (1-q)\kappa_2\right)}{2}.
\end{aligned}$$

The proof is completed. □

**Theorem 5.** *Let* $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ *be a function and* $0 < q < 1$. *Then we have*

$$\int_0^1 \mathcal{F}\left(\xi\kappa_2 + (1-\xi)\kappa_1\right) \ {}^1d_{\overline{q}}\xi = \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(t) \ {}^{\kappa_2}d_q^T t \ . \tag{3.3}$$

*Proof.* From definition of ${}^{\kappa_2}T_q$-integral, we have

$$\begin{aligned}
& \int_0^1 \mathcal{F}\left(\xi\kappa_2 + (1-\xi)\kappa_1\right) \ {}^1d_{\overline{q}}\xi \\
= \ & \frac{(1-q)(1-0)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}\left(\left[q^n 0 + (1-q^n)1\right]\kappa_2 + \left(1 - \left[q^n 0 + (1-q^n)1\right]\right)\kappa_1\right)\right. \\
& \left. -\mathcal{F}\left(0b + (1-0)\kappa_1\right)\right] \\
= \ & \frac{(1-q)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\kappa_1 + (1-q^n)\kappa_2\right) - \mathcal{F}(\kappa_1)\right] \\
= \ & \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(t) \ {}^{\kappa_2}d_q^T t \ .
\end{aligned}$$

The proof is completed. □

**Theorem 6.** *Let* $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ *be a continuous function. Then we have*

$$\int_{\varkappa}^{\kappa_2} {}^{\kappa_2}D_q \ \mathcal{F}(\xi) \ {}^{\kappa_2}d_q^T\xi = \frac{(1+q)\mathcal{F}(\kappa_2) - q\mathcal{F}(\varkappa) - \mathcal{F}(q\varkappa + (1-q)\kappa_2)}{2q} \tag{3.4}$$

*for* $\varkappa \in (\kappa_1, \kappa_2)$

*Proof.* From Definition 2, we have

$$ {}_{\kappa_2}D_q \ \mathcal{F}(\xi) = \frac{\mathcal{F}(q\xi + (1-q)\kappa_2) - \mathcal{F}(\xi)}{(1-q)(\kappa_2 - \xi)}. $$

By using Definition 6, we have

$$\begin{aligned}
&\int_{\varkappa}^{\kappa_2} {}_{\kappa_2}D_q \ \mathcal{F}(\xi) \ {}^{\kappa_2}d_q^T\xi \\
= &\int_{\varkappa}^{\kappa_2} \frac{\mathcal{F}(q\xi + (1-q)\kappa_2) - \mathcal{F}(\xi)}{(1-q)(\kappa_2 - \xi)} \ {}^{\kappa_2}d_q^T\xi \\
= &\frac{(1-q)(\kappa_2 - \varkappa)}{2q}\left[(1+q)\sum_{n=0}^{\infty}\frac{q^n\mathcal{F}\left(q^{n+1}\varkappa + \left(1-q^{n+1}\right)\kappa_2\right)}{(1-q)q^n(\kappa_2-\varkappa)} - \frac{\mathcal{F}(q\varkappa + (1-q)\kappa_2)}{(1-q)(\kappa_2-\varkappa)}\right] \\
&-\frac{(1-q)(\kappa_2 - \varkappa)}{2q}\left[(1+q)\sum_{n=0}^{\infty}\frac{q^n\mathcal{F}(q^n\varkappa + (1-q^n)\kappa_2)}{(1-q)q^n(\kappa_2-\varkappa)} - \frac{\mathcal{F}(\varkappa)}{(1-q)(\kappa_2-\varkappa)}\right] \\
= &\frac{1+q}{2q}\sum_{n=0}^{\infty}\mathcal{F}\left(q^{n+1}\varkappa + \left(1-q^{n+1}\right)\kappa_2\right) - \mathcal{F}(q^n\varkappa + (1-q^n)\kappa_2)] \\
&+\frac{1}{2q}\left[\mathcal{F}(q\varkappa + (1-q)\kappa_2) + \mathcal{F}(\varkappa)\right] \\
= &\frac{(1+q)\mathcal{F}(\kappa_2) - q\mathcal{F}(\varkappa) - \mathcal{F}(q\varkappa + (1-q)\kappa_2)}{2q}
\end{aligned}$$

The proof is complated. □

**Theorem 7.** *Assume that* $\mathcal{F}, g : [\kappa_1, \kappa_2] \to \mathbb{R}$ *are continuous functions. Then we have*

$$ \int_{\varkappa}^{\kappa_2} [\mathcal{F}(\xi) + g(\xi)] \ {}^{\kappa_2}d_q^T\xi = \int_{\varkappa}^{\kappa_2} \mathcal{F}(\xi) \ {}^{\kappa_2}d_q^T\xi + \int_{\varkappa}^{\kappa_2} g(\xi) \ {}^{\kappa_2}d_q^T\xi \tag{3.5} $$

*for* $\varkappa \in [\kappa_1, \kappa_2]$.

*Proof.* Using the definition of ${}^{\kappa_2}T_q$-integral, we can write that

$$\begin{aligned}
&\int_{\varkappa}^{\kappa_2} [\mathcal{F}(\xi) + g(\xi)] \ {}^{\kappa_2}d_q^T\xi \\
= &\frac{(1-q)(\kappa_2 - \varkappa)}{2q}\left\{(1+q)\sum_{n=0}^{\infty} q^n\left[\mathcal{F}(q^n\varkappa + (1-q^n)\kappa_2) + g(q^n\varkappa + (1-q^n)\kappa_2)\right]\right. \\
&\left.-\mathcal{F}(\varkappa) - g(\varkappa)\right\} \\
= &\frac{(1-q)(\kappa_2 - \varkappa)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\varkappa + (1-q^n)\kappa_2) - \mathcal{F}(\varkappa)\right] \\
&+\frac{(1-q)(\kappa_2 - \varkappa)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n g(q^n\varkappa + (1-q^n)\kappa_2) - g(\varkappa)\right] \\
= &\int_{\varkappa}^{\kappa_2} \mathcal{F}(\xi) \ {}^{\kappa_2}d_q^T\xi + \int_{\varkappa}^{\kappa_2} g(\xi) \ {}^{\kappa_2}d_q^T\xi
\end{aligned}$$

which finishes proof. □

**Theorem 8.** *Assume that $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ are continuous functions and $\alpha\in\mathbb{R}$. Then*

$$\int_{\varkappa}^{\kappa_2}(\alpha\mathcal{F})(\xi)\ {}^{\kappa_2}d_q^T\xi\ =\alpha\int_{\varkappa}^{\kappa_2}\mathcal{F}(\xi)\ {}^{\kappa_2}d_q^T\xi \tag{3.6}$$

*for $\varkappa\in[\kappa_1,\kappa_2]$.*

*Proof.* Bt the definition of ${}^{\kappa_2}T_q$-integral, we have

$$\begin{aligned}\int_{\varkappa}^{\kappa_2}(\alpha\mathcal{F})(\xi)\ {}^{\kappa_2}d_q^T\xi &= \frac{(1-q)(\kappa_2-\varkappa)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^n(\alpha\mathcal{F})(q^n\varkappa+(1-q^n)\kappa_2)-(\alpha\mathcal{F})(\varkappa)\right]\\ &= \alpha\frac{(1-q)(\kappa_2-\varkappa)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\varkappa+(1-q^n)\kappa_2)-\mathcal{F}(\varkappa)\right]\\ &= \alpha\int_{\varkappa}^{\kappa_2}\mathcal{F}(\xi)\ {}^{\kappa_2}d_q^T\xi\ .\end{aligned}$$

□

**Theorem 9.** *Assume that $\mathcal{F},g:[\kappa_1,\kappa_2]\to\mathbb{R}$ are continuous functions. Then we have*

$$\begin{aligned}&\int_{\varkappa}^{\kappa_2}\mathcal{F}(\xi)\ {}^{\kappa_2}D_q\ g(\xi)\ {}^{\kappa_2}d_q^T\xi\\ &= \frac{q\mathcal{F}(\xi)g(\xi)+\mathcal{F}(q\xi+(1-q)\kappa_2)g(q\xi+(1-q)\kappa_2)}{2q}\Big|_{\varkappa}^{\kappa_2}\\ &\quad-\int_{\varkappa}^{\kappa_2}g(q\xi+(1-q)\kappa_2)\ {}^{\kappa_2}D_q\ \mathcal{F}(\xi)\ {}^{\kappa_2}d_q^T\xi\end{aligned} \tag{3.7}$$

*for $\varkappa\in[\kappa_1,\kappa_2]$.*

*Proof.* Using Definition 2, we get

$$\begin{aligned}&{}^{\kappa_2}D_q\ (\mathcal{F}(\xi)g(\xi))\\ &= \frac{\mathcal{F}(q\xi+(1-q)\kappa_2)g(q\xi+(1-q)\kappa_2)-\mathcal{F}(\xi)g(\xi)}{(1-q)(\kappa_2-\xi)}\\ &= \mathcal{F}(\xi)\frac{g(q\xi+(1-q)\kappa_2)-g(\xi)}{(1-q)(\kappa_2-\xi)}+g(q\xi+(1-q)\kappa_2)\frac{\mathcal{F}(q\xi+(1-q)\kappa_2)-\mathcal{F}(\xi)}{(1-q)(\kappa_2-\xi)}\\ &= \mathcal{F}(\xi)\ {}^{\kappa_2}D_q\ g(\xi)+g(q\xi+(1-q)\kappa_2)\ {}^{\kappa_2}D_q\ \mathcal{F}(\xi).\end{aligned} \tag{3.8}$$

By taking ${}^{\kappa_2}T_q$-integral of the equality (3.8), we get

$$\begin{aligned}&\int_{\varkappa}^{\kappa_2}\ {}^{\kappa_2}D_q\ (\mathcal{F}(\xi)g(\xi))\ {}^{\kappa_2}d_q^T\xi\\ &= \int_{\varkappa}^{\kappa_2}\mathcal{F}(\xi)\ {}^{\kappa_2}D_q\ g(\xi)\ {}^{\kappa_2}d_q^T\xi\ +\int_{\varkappa}^{\kappa_2}g(q\xi+(1-q)\kappa_2)\ {}^{\kappa_2}D_q\ \mathcal{F}(\xi)\ {}^{\kappa_2}d_q^T\xi\end{aligned} \tag{3.9}$$

By applying Theorem 2.4, we have

$$\begin{aligned}(3.10)\qquad &\int_{\varkappa}^{\kappa_2} {}^{\kappa_2}D_q \ \left(\mathcal{F}(\xi)\, g(\xi)\right) \ {}^{\kappa_2}d_q^T\xi \\ &= \frac{(1+q)\,\mathcal{F}(\kappa_2)\,g(\kappa_2) - q\mathcal{F}(\xi)\,g(\xi) - \mathcal{F}(q\xi+(1-q)\,\kappa_2)\,g(q\xi+(1-q)\,\kappa_2)}{2q} \\ &= \frac{q\mathcal{F}(\xi)\,g(\xi) + \mathcal{F}(q\xi+(1-q)\,\kappa_2)\,g(q\xi+(1-q)\,\kappa_2)}{2q}\Bigg|_{\varkappa}^{\kappa_2}.\end{aligned}$$

From the equalities (3.9) and (3.10), we obtain

$$\begin{aligned}\int_{\varkappa}^{\kappa_2} \mathcal{F}(\xi) \ {}^{\kappa_2}D_q \ g(\xi) \ {}^{\kappa_2}d_q^T\xi &= \frac{q\mathcal{F}(\xi)\,g(\xi) + \mathcal{F}(q\xi+(1-q)\,\kappa_2)\,g(q\xi+(1-q)\,\kappa_2)}{2q}\Bigg|_{\varkappa}^{\kappa_2} \\ &\quad - \int_{\varkappa}^{\kappa_2} g(q\xi+(1-q)\,\kappa_2) \ {}^{\kappa_2}D_q \ \mathcal{F}(\xi) \ {}^{\kappa_2}d_q^T\xi\ .\end{aligned}$$

This completes the proof. □

**Theorem 10.** *Assume that $\mathcal{F}, g : [\kappa_1, \kappa_2] \to \mathbb{R}$ are continuous functions and $\mathcal{F}(\xi) \le g(\xi)$ for all $\xi \in [\varkappa, \kappa_2]$. Then we have*

$$\int_{\varkappa}^{\kappa_2} \mathcal{F}(\xi) \ {}^{\kappa_2}d_q^T\xi \ \le \int_{\varkappa}^{\kappa_2} g(\xi) \ {}^{\kappa_2}d_q^T\xi$$

*for $\varkappa \in [\kappa_1, \kappa_2]$.*

*Proof.* Bt the definition of ${}^{\kappa_2}T_q$-integral, we have

$$\begin{aligned}\int_{\varkappa}^{\kappa_2} \mathcal{F}(\xi) \ {}^{\kappa_2}d_q^T\xi &= \frac{(1-q)(\kappa_2-\varkappa)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\varkappa+(1-q^n)\,\kappa_2) - \mathcal{F}(\varkappa)\right] \\ &= \frac{(1-q)(\kappa_2-\varkappa)}{2q}\left[(1+q)\sum_{n=1}^{\infty} q^n\mathcal{F}(q^n\varkappa+(1-q^n)\,\kappa_2) + q\mathcal{F}(\varkappa)\right] \\ &\le \frac{(1-q)(\kappa_2-\varkappa)}{2q}\left[(1+q)\sum_{n=1}^{\infty} q^n g(q^n\varkappa+(1-q^n)\,\kappa_2) + qg(\varkappa)\right] \\ &= \frac{(1-q)(\kappa_2-\varkappa)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n g(q^n\varkappa+(1-q^n)\,\kappa_2) - g(\varkappa)\right] \\ &= \int_{\varkappa}^{\kappa_2} g(\xi) \ {}^{\kappa_2}d_q^T\xi\ .\end{aligned}$$

□

**Proposition 1.** *For $\alpha \in \mathbb{R}\setminus\{-1\}$, we have the following equality*

$$(3.11)\qquad \int_{x}^{b} (b-s)^{\alpha} \ {}^{b}d_{\overline{q}}s \ = \frac{1+q^{\alpha}}{2\,[\alpha+1]_q}\,(b-x)^{\alpha+1}.$$

*Proof.* Using the definition of ${}^{\kappa_2}T_q$-integral, we have

$$\begin{aligned}
\int_{\varkappa}^{\kappa_2} (\kappa_2-\xi)^{\alpha} \ {}^{\kappa_2}d_q^T\xi &= \frac{(1-q)(\kappa_2-\varkappa)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n\left(\kappa_2-\left(q^n\varkappa+(1-q^n)\kappa_2\right)\right)^{\alpha}-(\kappa_2-\varkappa)^{\alpha}\right]\\
&= \frac{(1-q)(\varkappa-\kappa_1)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n\left(q^n(\kappa_2-\varkappa)\right)^{\alpha}-(\kappa_2-\varkappa)^{\alpha}\right]\\
&= \frac{(1-q)(\kappa_2-\varkappa)}{2q}\left[(1+q)(\kappa_2-\varkappa)^{\alpha}\sum_{n=0}^{\infty}\left(q^{\alpha+1}\right)^n-(\kappa_2-\varkappa)^{\alpha}\right]\\
&= \frac{(1-q)(\kappa_2-\varkappa)^{\alpha+1}}{2q}\left[\frac{(1+q)}{1-q^{\alpha+1}}-1\right]\\
&= \frac{1-q}{1-q^{\alpha+1}}\frac{1+q^{\alpha}}{2}(\kappa_2-\varkappa)^{\alpha+1}\\
&= \frac{1}{[\alpha+1]_q}\frac{1+q^{\alpha}}{2}(\kappa_2-\varkappa)^{\alpha+1}.
\end{aligned}$$

The proof is completed. □

## 4. Hermite-Hadamard Inequalities for ${}^{\kappa_2}T_q$-integral

In this section, we present some Hermite-Hadamard type inequalities for ${}^{\kappa_2}T_q$-integral by utulizing convex functions .

**Theorem 11.** *Let $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a convex continuous function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then we have*

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\le\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q^T\varkappa\ \le\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2} \tag{4.1}$$

*Proof.* Since $\mathcal{F}$ is differentiable function on $[\kappa_1,\kappa_2]$, there is a tangent line for the function $\mathcal{F}$ at the point $\frac{\kappa_1+\kappa_2}{2}\in(\kappa_1,\kappa_2)$. This tangent line can be expressed as a function $\Psi_1(\varkappa)=\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathcal{F}'\left(\frac{\kappa_1+\kappa_2}{2}\right)\left(\varkappa-\frac{\kappa_1+\kappa_2}{2}\right)$.

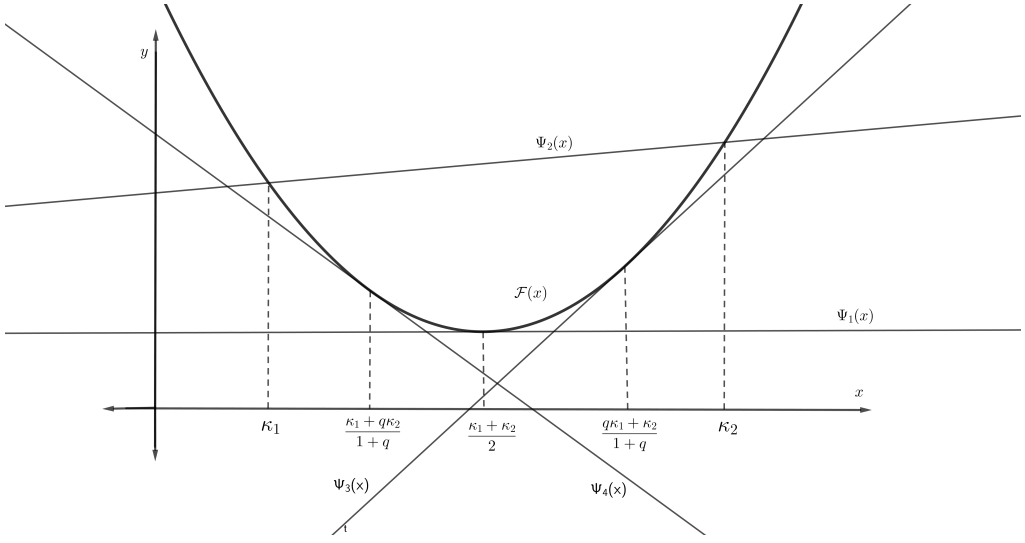


FIGURE 2. Graphs of a convex function $\mathcal{F}$ and some tangent lines

Since $\mathcal{F}$ is a convex function on $[\kappa_1, \kappa_2]$, then we have the following inequality

$$\Psi_1(\varkappa) = \mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + \mathcal{F}'\left(\frac{\kappa_1+\kappa_2}{2}\right)\left(\varkappa - \frac{\kappa_1+\kappa_2}{2}\right) \leq \mathcal{F}(\varkappa) \tag{4.2}$$

for all $\varkappa \in [\kappa_1, \kappa_2]$ (see Figure 2). From Theorem 10, we have

$$\int_{\kappa_1}^{\kappa_2} \Psi_1(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \ \leq \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \ .$$

By Definition 6, we have

$$\begin{aligned}
&\int_{\kappa_1}^{\kappa_2} \Psi_1(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \\
&= \int_{\kappa_1}^{\kappa_2} \left[\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + \mathcal{F}'\left(\frac{\kappa_1+\kappa_2}{2}\right)\left(\varkappa - \frac{\kappa_1+\kappa_2}{2}\right)\right] \ {}^{\kappa_2}d_q^T\varkappa \\
&= (\kappa_2-\kappa_1)\,\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \mathcal{F}'\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{\kappa_1}^{\kappa_2}\left(\kappa_2 - \varkappa + \frac{\kappa_1-\kappa_2}{2}\right) \ {}^{\kappa_2}d_q^T\varkappa \\
&= (\kappa_2-\kappa_1)\,\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \mathcal{F}'\left(\frac{\kappa_1+\kappa_2}{2}\right) \\
&\quad \times \left[\left(\frac{1-q}{1-q^2}\right)\left(\frac{1+q}{2}\right)(\kappa_2-\varkappa)^2\Big|_{\kappa_1}^{\kappa_2} + \frac{\kappa_1-\kappa_2}{2}(\kappa_2-\kappa_1)\right] \\
&= (\kappa_2-\kappa_1)\,\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \mathcal{F}'\left(\frac{\kappa_1+\kappa_2}{2}\right)\left[\frac{(\kappa_2-\kappa_1)^2}{2} - \frac{(\kappa_2-\kappa_1)^2}{2}\right] \\
&= (\kappa_2-\kappa_1)\,\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)
\end{aligned}$$

This gives the proof of first inequality in (4.1).

On the other hand, we have the function $\Psi_2(\varkappa) = \mathcal{F}(\kappa_2) + \frac{\mathcal{F}(\kappa_2)-\mathcal{F}(\kappa_1)}{\kappa_2-\kappa_1}(\varkappa-\kappa_2)$ (see Figure 2). Since $\mathcal{F}$ is convex function on $[\kappa_1, \kappa_2]$, we have the inequality

$$\mathcal{F}(\varkappa) \leq \Psi_2(\varkappa)$$

and thus, by Theorem 10, we get

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \ \leq \int_{\kappa_1}^{\kappa_2} \Psi_2(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa$$

for all $\varkappa \in [\kappa_1, \kappa_2]$. By definition of ${}^{\kappa_2}T_q$-integral, we have

$$\begin{aligned}
&\int_{\kappa_1}^{\kappa_2} \Psi_2(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \\
&= \int_{\kappa_1}^{\kappa_2} \left(\mathcal{F}(\kappa_2) + \frac{\mathcal{F}(\kappa_2)-\mathcal{F}(\kappa_1)}{\kappa_2-\kappa_1}(\varkappa-\kappa_2)\right) \ {}^{\kappa_2}d_q^T\varkappa \\
&= (\kappa_2-\kappa_1)\mathcal{F}(\kappa_2) - \frac{\mathcal{F}(\kappa_2)-\mathcal{F}(\kappa_1)}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}(\kappa_2-\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \\
&= (\kappa_2-\kappa_1)\mathcal{F}(\kappa_2) - \frac{\mathcal{F}(\kappa_2)-\mathcal{F}(\kappa_1)}{\kappa_2-\kappa_1}\left(\frac{1-q}{1-q^2}\right)\left(\frac{1+q}{2}\right)(\kappa_2-\kappa_1)^2 \\
&= (\kappa_2-\kappa_1)\mathcal{F}(\kappa_2) - \frac{\mathcal{F}(\kappa_2)-\mathcal{F}(\kappa_1)}{\kappa_2-\kappa_1}\frac{(\kappa_2-\kappa_1)^2}{2} \\
&= (\kappa_2-\kappa_1)\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}.
\end{aligned}$$

The proof is completed. □

**Remark 1.** *In Theorem 4.1, if we take the limit $q \to 1^-$, we recapture the classical Hermite-Hadamard inequality for convex function.*

**Theorem 12.** *Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex differentiable function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then we have*

(4.3)

$$\mathcal{F}\left(\frac{q\kappa_1+\kappa_2}{1+q}\right) + \frac{q-1}{1+q}\frac{(\kappa_2-\kappa_1)}{2}\mathcal{F}'\left(\frac{q\kappa_1+\kappa_2}{1+q}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \ \leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}.$$

*Proof.* Since $\mathcal{F}$ is differentiable function on $[\kappa_1, \kappa_2]$, there is a tangent line for the function $\mathcal{F}$ at the point $\frac{q\kappa_1+\kappa_2}{1+q} \in (\kappa_1, \kappa_2)$. This tangent line can be expressed as a function $\Psi_3(\varkappa) = \mathcal{F}\left(\frac{q\kappa_1+\kappa_2}{1+q}\right) + \mathcal{F}'\left(\frac{q\kappa_1+\kappa_2}{1+q}\right)\left(\varkappa - \frac{q\kappa_1+\kappa_2}{1+q}\right)$. Since $\mathcal{F}$ is a convex function on $[\kappa_1, \kappa_2]$, then we have the following inequality

$$\Psi_3(\varkappa) \leq \mathcal{F}(\varkappa) \tag{4.4}$$

for all $\varkappa \in [\kappa_1, \kappa_2]$ (see Figure 2). From Theorem 10, we have

$$\int_{\kappa_1}^{\kappa_2}\Psi_3(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \ \leq \int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \ .$$

By definition of ${}^{\kappa_2}T_q$-integral, we get

$$\begin{aligned}
&\int_{\kappa_1}^{\kappa_2} \Psi_3(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \\
&= \int_{\kappa_1}^{\kappa_2} \left[\mathcal{F}\left(\frac{q\kappa_1+\kappa_2}{1+q}\right) + \mathcal{F}'\left(\frac{q\kappa_1+\kappa_2}{1+q}\right)\left(\varkappa - \frac{q\kappa_1+\kappa_2}{1+q}\right)\right] \ {}^{\kappa_2}d_q^T\varkappa \\
&= (\kappa_2-\kappa_1)\mathcal{F}\left(\frac{q\kappa_1+\kappa_2}{1+q}\right) - \mathcal{F}'\left(\frac{q\kappa_1+\kappa_2}{1+q}\right)\left[\int_{\kappa_1}^{\kappa_2}\left(\kappa_2 - \varkappa + q\frac{\kappa_1-\kappa_2}{1+q}\right) \ {}^{\kappa_2}d_q^T\varkappa\right] \\
&= (\kappa_2-\kappa_1)\mathcal{F}\left(\frac{q\kappa_1+\kappa_2}{1+q}\right) - \mathcal{F}'\left(\frac{q\kappa_1+\kappa_2}{1+q}\right)\left[\left(\frac{1-q}{1-q^2}\right)\left(\frac{1+q}{2}\right)(\kappa_2-\kappa_1)^2 + \frac{\kappa_1-\kappa_2}{1+q}q(\kappa_2-\kappa_1)\right] \\
&= (\kappa_2-\kappa_1)\mathcal{F}\left(\frac{q\kappa_1+\kappa_2}{1+q}\right) - \mathcal{F}'\left(\frac{q\kappa_1+\kappa_2}{1+q}\right)\left[\frac{(\kappa_2-\kappa_1)^2}{2} - \frac{q(\kappa_2-\kappa_1)^2}{1+q}\right]
\end{aligned}$$

which completes the proof. □

**Theorem 13.** *Let $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a convex differentiable function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then we have*

(4.5)
$$\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{1+q}\right) + \frac{1-q}{1+q}\frac{(\kappa_2-\kappa_1)}{2}\mathcal{F}'\left(\frac{\kappa_1+q\kappa_2}{1+q}\right) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \ \le \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}.$$

*Proof.* Since $\mathcal{F}$ is differentiable function on $[\kappa_1,\kappa_2]$, there is a tangent line for the function $\mathcal{F}$ at the point $\frac{\kappa_1+q\kappa_2}{1+q}\in(\kappa_1,\kappa_2)$. This tangent line can be expressed as a function $\Psi_4(\varkappa)=\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{1+q}\right)+\mathcal{F}'\left(\frac{\kappa_1+q\kappa_2}{1+q}\right)\left(\varkappa-\frac{\kappa_1+q\kappa_2}{1+q}\right)$. Since $\mathcal{F}$ is a convex function on $[\kappa_1,\kappa_2]$, then we have the following inequality

$$\Psi_4(\varkappa)\le\mathcal{F}(\varkappa) \tag{4.6}$$

for all $\varkappa\in[\kappa_1,\kappa_2]$ (see Figure 2). By Theorem 10, we have

$$\int_{\kappa_1}^{\kappa_2}\Psi_4(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \ \le \int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \ .$$

By definition of ${}^{\kappa_2}T_q$-integral, we get

$$\begin{aligned}
&\int_{\kappa_1}^{\kappa_2}\Psi_4(\varkappa) \ {}^{\kappa_2}d_q^T\varkappa \\
&= \int_{\kappa_1}^{\kappa_2}\left[\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{1+q}\right)+\mathcal{F}'\left(\frac{\kappa_1+q\kappa_2}{1+q}\right)\left(\varkappa-\frac{\kappa_1+q\kappa_2}{1+q}\right)\right] \ {}^{\kappa_2}d_q^T\varkappa \\
&= (\kappa_2-\kappa_1)\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{1+q}\right) - \mathcal{F}'\left(\frac{\kappa_1+q\kappa_2}{1+q}\right)\int_{\kappa_1}^{\kappa_2}\left(\kappa_2-\varkappa+\frac{\kappa_1-\kappa_2}{1+q}\right) \ {}^{\kappa_2}d_q^T\varkappa \\
&= (\kappa_2-\kappa_1)\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{1+q}\right) - \mathcal{F}'\left(\frac{\kappa_1+q\kappa_2}{1+q}\right)\left[\left(\frac{1-q}{1-q^2}\right)\left(\frac{1+q}{2}\right)(\kappa_2-\kappa_1)^2 + (\kappa_2-\kappa_1)\frac{\kappa_1-\kappa_2}{1+q}\right] \\
&= (\kappa_2-\kappa_1)\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{1+q}\right) - \mathcal{F}'\left(\frac{\kappa_1+q\kappa_2}{1+q}\right)\left[\frac{(\kappa_2-\kappa_1)^2}{2} - \frac{(\kappa_2-\kappa_1)^2}{1+q}\right].
\end{aligned}$$

This gives the proof of theorem. □

**Theorem 14.** *Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex differentiable function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then we have*

$$\max\{I_1, I_2, I_3\} \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \ {}^{\kappa_2} d_q^T \varkappa \ \leq \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}. \tag{4.7}$$

*where*

$$I_1 = \mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right),$$

$$I_2 = \mathcal{F}\left(\frac{q\kappa_1 + \kappa_2}{1+q}\right) + \frac{q-1}{1+q}\frac{(\kappa_2 - \kappa_1)}{2}\mathcal{F}'\left(\frac{q\kappa_1 + \kappa_2}{1+q}\right)$$

$$I_3 = \mathcal{F}\left(\frac{\kappa_1 + q\kappa_2}{1+q}\right) + \frac{1-q}{1+q}\frac{(\kappa_2 - \kappa_1)}{2}\mathcal{F}'\left(\frac{\kappa_1 + q\kappa_2}{1+q}\right).$$

*Proof.* A combination of (4.1), (4.3), and (4.5) gives (4.7) and the proof is completed. □

## 5. Conclusion

In this article, we proved a new idea of quantum integrals which is called ${}^{\kappa_2}T_q$-integral. By using this idea, we proved several properties for quantum integrals. Further, we presented several Hermite-Hadamard type ${}^{\kappa_2}T_q$-integral inequalities within class of convexity. It is also shown that some classical results can be obtained by the results presented in the current research by taking the limit $q \to 1^-$. It will be an interesting problem to prove similar inequalities for the functions of two variables.

## References


[1] N. Alp, M. Z. Sarikaya, M. Kunt and I. Iscan, *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, Journal of King Saud University–Science (2018) 30, 193–203.

[2] N. Alp and M. Z. Sariakaya, *A new definition and properties of quantum integral which calls $\overline{q}$-integral*. Konuralp Journal of Mathematics, (2017), 5(2), 146-159.

[3] N. Alp and M. Z. Sarikaya, *Hermite Hadamard's type inequalities for co-ordinated convex functions on quantum integral,* Submitted. https://www.researchgate.net/publication/329814922

[4] S. Bermudo, P. Kórus and J. N. Valdés, *On q-Hermite–Hadamard inequalities for general convex functions*, Acta Mathematica Hungarica, 1-11, 2020.

[5] S. Belarbi and Z. Dahmani, *On some new fractional integral inequalities,* JIPAM. J. Inequal. Pure Appl. Math. 2009., 10: Article ID 86.

[6] Z. Dahmani, *New inequalities in fractional integrals,* Int. J. Nonlinear Sci. 2010, 9: 493–497.

[7] T. Ernst, A *Comprehensive Treatment of q-Calculus*, Springer Basel (2012).

[8] T. Ernst, *A method for q-calculus*. J. Nonlinear Math. Phys. 10 (4), 487–525 (2003).

[9] T. Ernst, *The History Of Q-Calculus And New Method. Sweden: Department of Mathematics*, Uppsala University, 2000.

[10] H. Gauchman, *Integral inequalities in q-calculus,* Comput. Math. Appl. 2004, 47: 281–300. 10.1016/S0898-1221(04)90025-9.

[11] J. Hadamard, *Etude sur les propri´et´es des fonctions enti´eres et en particulier dune fonction consider´ee par Riemann*, J. Math. Pures Appl. 58 (1893) 171–215.

[12] F. H. Jackson, *On a q-definite integrals*, Quarterly J. Pure Appl. Math. 41 (1910) 193-203.

[13] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2002.

[14] M.A. Noor, K.I. Noor and M.U. Awan, *Some Quantum estimates for Hermite–Hadamard inequalities,* Appl. Math. Comput. 251, 675–679 (2015).

[15] M. A. Noor, K. I. Noor, M. U. Awan, *Some quantum estimates for Hermite-Hadamard inequalities*, Appl. Math. Comput. 251 (2015) 675–679.

[16] M. A. Noor, K. I. Noor, M. U. Awan, *Some quantum integral inequalities via preinvex function*s, Appl. Math. Comput. 269 (2015) 242–251.

[17] M. Noor, K. Noor, and M. Awan, Quantum ostrowski inequalities for $q$-differentiable convex functions, Journal of Mathematical Inequalities, 2016.

[18] H. Ogunmez and U.M. Ozkan, *Fractional quantum integral inequalities*, J. Inequal. Appl. 2011., 2011: Article ID 787939

[19] W. Sudsutad and S. K. Ntouyas, J. Tariboon, Q*uantum integral inequalities for convex functions,* Jour. Math Ineq. Volume 9, Number 3 (2015), 781–793.

[20] W. Sudsutad, S. K. Ntouyas, J. Tariboon, *Quantum integral inequalities for convex functions*, J. Math. Inequal. 9 (3) (2015) 781–793.

[21] J. Tariboon and S. K. Ntouyas, *Quantum integral inequalities on finite intervals*, J. Inequal. Appl. 2014, 2014:121.

[22] J. Tariboon, S. K. Ntouyas, *Quantum calculus on finite intervals and applications to impulsive difference equations*, Adv. Difference Equ. 282 (2013) 1-19.
[23] H. Zhuang, W. Liu, J. Park, *Some quantum estimates of Hermite-Hadmard inequalities for quasi-convex functions*, Mathematics 2019, 7, 152.

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