A new approach for the generalized fractional Casson fluid model with Newtonian heating described by the modified Riemann-Liouville fractional operator

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Abstract

This research is about the transfer of heat of a generalized fractional Casson fluid on an unsteady boundary layer which is passing through an infinite oscillating plate, in vertical direction combined with the Newtonian heating. The results are obtained by using modified Riemann-Liouville fractional derivative. The present fluid model, starts with the governing equations which are then converted to a system of partial differential equations(linear) by using some suitable non-dimensional variables. Using the method of integral balance and the Laplace transform technique, an analytical solution is obtained. The velocity and temperature expressions are derived and the effects of modelling parameters re shown in tables and graphs to validate the obtained theoretical results.

Keywords: Casson fluid model, integral balance method, fractional heat equation, generalized fractional derivative, Newtonian heating, modified fractional derivative

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1 Introduction

Many scientists extensively analyse the boundary layer flow of viscous and non-Newtonian fluids due to its significance in continuous casting, polymer extrusion, paper production, aerodynamic extrusion of plastic sheet and many more. The study of unsteady boundary layer is valuable in various physical problems such as flow over a helicopter in translation motion, blades of turbines and compressors, aerodynamic surfaces of vehicles in manned flight, etc. The unsteadiness in the flow field is induced either by time dependent motion of the external stream or by impulsive motion of the external stream.

When the motion of fluid over a body is established impulsively, the inviscid flow is generated instantaneously over the body but in the mean time the viscous layer is developed slowly near the body and it become as fully developed steady state viscous flow after a certain period of time. Velocity, temperature and concentration are studied for different parameters like magnetic field, Prandtl number, chemical reaction and heat source. Convective heat transfer interprets a major role during the handling and processing of non-Newtonian fluid flows. Mechanics of non-Newtonian fluid flows gives a special inspiration to engineers, chemists, physicists and mathematicians.

The non-Newtonian fluids are generally categorized into three cases, namely differential, rate, and integral. The straight forward subclass of the second type of fluids is the Maxwell model which can predict stress relaxation. This rheological model eliminates the complicated effects of shear-dependent viscosity from any boundary layer analysis. Another type of non-Newtonian fluid is known as Casson fluid, which indicates yield stress. Casson is a shear thinning liquid which is assumed to have an infinite viscosity at zero rate of shear. Below the yield stress there is no flow, and at an infinite rate of shear, the viscosity is zero, that is when a shear stress is less than the yield stress, the fluid performs like a solid, but when a shear stress is greater than the yield stress, the fluid starts moving Mathematicians as well as researchers working in the medical field are broadly working on Casson fluid model. Some well known examples of the Casson fluid are honey, soup, jelly, tomato sauce, etc. In the literature, the Casson model is erratically claimed to fit rheological data that are preferable than the general visco-plastic models for many materials. Human blood is the best example of Casson fluid due to the existence of several substances such as protein, fibrinogen and globulin in aqueous base plasma and human red blood cells can form a train structure, known as stacks or aggregations. If it behaves like a plastic solid, then there exists a yield stress that can be determined with the constant yield stress in Cassons fluid.

On the other hand, in view of the applications, Newtonian fluids follow a linear relationship between the stress and the rate of strain that is limited. Over the range of shear stresses and shear rates that we come across in our day-to-day life, water, air, alcohol, glycerol, thin motor oil, etc are some examples of Newtonian fluids . Single-phase fluids consist of small molecules that are generally Newtonian. The process of heat transfer into the convective fluid through bounding surface having finite heat capacity is called as Newtonian heating. This type of phenomenon exists in the system of conventional flows if the heat is injected by solar radiation. In all the above studies, the flow is driven by a prescribed surface temperature or by heat flux. Here a new different driving mechanism for unsteady free convection along a vertical surface is considered where the flow is set up by Newtonian heating from the surface. The characteristics of heat transfer depend on the thermal boundary conditions. In general, the heat is transferred from one object to another via three processes, namely conduction, radiation, and convection.

The interface temperature is initially unknown but relate to the intrinsic properties of the system, such as the thermal conductivity of the solid and fluid respectively. The heat transfer rate with a finite heat capacity from the bounding surface is directly proportional to the local surface temperature and is generally referred as the conjugate of the convective flow. This type of formation occurs in many cases of engineering, such as heat exchangers in which (i) the conduction in solid tube wall is highly influenced by the convection in the flow of fluid; (ii) for conjugate heat transfer around fins where the conduction within the fin and the convection in the fluid encircling it must be concurrently analysed in order to attain the vital design information; and (iii) convective flows set up by solar radiation where the bounding surfaces absorb heat. Hence the standard assumption of no interaction of conduction-convection effects is not normally feasible and it might be assumed in many practical engineering applications when evaluating the conjugate heat transfer processes.

2 Review of Literature

The Newtonian heating condition has been recently applied in convective heat transfer. Merkin [23] was the first to examine the free-convection boundary layer over a vertical flat plate involved in a viscous fluid. Hussanan et al, [11, 12] started studying the Newtonian heating of the natural convection flow passing through an oscillating plate and unsteady flow and heat transfer of a Casson fluid on oscillating vertical plate. From the above literature, it should be noted that there are several contributions on Newtonian heating [9,13,19]. On the other side, many authors have discussed various methods to handle linear and non-linear fractional differential equations which have great influence in scientific and technological disciplines.

The flow of Casson fluids (for example, drilling muds, liquid chocolates, clay coatings and other suspensions, certain oils and greases, polymer melts, blood, and emulsions) in the presence of heat transfer, is an emerging area of research due to its relevance to the optimized processing of chocolates, ice-creams, and other food-stuffs [6]. Boundary layer flow and heat transfer of non-Newtonian fluids are analysed in ref. [8]. There are so many techniques available in the literature such as differential transform method, homotopy perturbation method, and variational iteration method for identifying the solutions of extensive applications in various areas of bio-chemical, rheology, physics and petroleum industries [1–3, 5, 10, 29]. Because of these applications, the perception and the study of non-Newtonian fluids become a fascinating topic of current research in this field. In all the above cases, the solutions of generalized fractional Casson fluid are determined by either using approximate or any numerical methods. Only very few studies are available in which the analytical solutions of generalized fractional Casson fluid are obtained. These solutions are rare when the fluid in free convection flows with constant wall temperature. Representative studies dealing with the non-Newtonian model can be found in [4, 21, 22, 25, 27, 28, 30–34].

The concept of fFractional differential equations is considered as an alternative tool to model nonlinear differential equations. The history of fractional calculus is mainly based on the article by, Miller and Ross [24] and recent review articles. Podlubny [26] unites both the derivative and the integral representation in a single expression and has given a connection between the Riemann-Liouville fractional derivative and the Grunwald-Letnikov fractional derivative. Then in 1967 M. Caputo [6] introduced a few format of fractional derivative which is in contrast to the Riemann-Liouville fractional derivative; while solving differential equations using Caputo's definition, it is not necessary to define the fractional order initial conditions. Since then, many researchers have tried to extend and find new definitions for the fractional derivative. Kilbas [20] provided an encyclopaedic mathematical treatment of fractional integrals and derivatives.

The above generalized Casson fluid model along with Jumaries modified Riemann-Liouville fractional derivative, which was originated by Jumarie for the first time [14–18], has some advantages compared with Riemann-Liouville and Caputo derivatives since the Riemann-Liouville derivative of a constant is not zero and the Caputo derivative is defined only for differentiable functions, for instance, some functions that have no first order derivatives but have fractional derivatives of all orders less than one. To overcome such situations, modified Riemann-Liouville derivative might be applied. The (Jumarie)modified Riemann-Liouville fractional derivative of a constant is equal to zero and it is defined for any continuous (non-differentiable) functions.

To the best knowledge of the author, the study of unsteady boundary layer flow of a Casson fluid past an oscillating vertical plate with constant wall temperature with Jumaries modified Riemann-Liouville derivative has not been obtained yet. Exact solutions are extracted by using the Laplace transform technique. Graphical results are presented and discussed for various physical parameters. The solutions obtained in this study are essential for the fundamental flow situations and for describing the flow physics in detail as well as for validation of other solutions obtained via approximate and numerical techniques.

3 Basic definitions and preliminaries

This section focus on the tools in fractional calculus; that is the fractional differential and integral operators. The definitions of fractional integral, Riemann-Liouville derivative, Caputo derivative, Modified Riemann-Liouville (Jumarie) derivative and their associated Laplace transforms are discussed.

Definition 3.1 *[26] The fractional order integral of the function f on a given interval $[a, b]$ of order $\alpha \in \mathbb{R}^+$ is defined by*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

provided that the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(.)$ is the gamma function.

Definition 3.2 *[26] The α^{th} Riemann-Lioville fractional derivative of the function f on a given interval $[a, b]$ is defined by*

$$(D_{a,t}^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 3.3 *[26] The Caputo fractional derivative of order α, for the function f on a given interval $[a,b]$ is defined by*

$$({}^{c}D_{a,t}^{\alpha}f)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds,$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

In the definition of Caputo, first differentiate the function $f(t)$ by $n-$ times and then integrate it by $n-\alpha$ times. The disadvantage of this derivative is that the function $f(t)$, must be differentiable by $n-$ times then only the $\alpha-$th order derivative will exist, where $n-1\leq\alpha<n$. If the function is not differentiable then the Caputo fractional derivative will not be applicable. There are two main advantages of this derivative, which are (i) fractional derivative of a constant is zero (ii) the Caputo type fractional differential equation has initial conditions of classical derivative type but the Riemann-Liouville type differential equations has initial conditions of fractional type i.e $\lim\limits_{t\to a}{}_aD_t^{\alpha-1}[f(t)]=b$.

With this, a fractional differential equation together with Riemann-Liouville fractional derivatives require concept of fractional initial stats, are very tough to interpret physically. To overcome the fractional derivative of a constant, non-zero, another modification of the definition of left Riemann-Liouville type fractional derivative of the function $f(t)$, in the interval $[a,b]$ was proposed by Jumarie in the following form.

Definition 3.4 *[15] The Modified Riemann-Liouville (Jumarie) fractional derivative of order α for the function f on a given interval $[a,b]$ is defined by*

$$({}^{J}D_{a,t}^{\alpha}f)(t)=\begin{cases}\dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_a^t(t-s)^{-\alpha-1}ds, & \alpha<0.\\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\displaystyle\int_a^t(t-s)^{-\alpha}[f(s)-f(a)]ds, & 0<\alpha<1.\\ (f^{\alpha-n}(t))^{(n)}, & n\leq\alpha<n+1.\end{cases}$$

We consider that $f(t)-f(a)=0$, for $t<a$. In the definition of Jumarie, the first expression is just fractional integration; the second line is Riemann-Liouville derivative of order $0<\alpha<1$ of offset function that is $f(t)-f(a)$. For $\alpha>1$, we use the third line; that is first we differentiate the offset function with order $0<(\alpha-n)<1$, by the formula of second line, and then apply whole n order differential to it. Here we choose integer n, just less than the real number α; that is $n\leq\alpha<n+1$.

Remark 3.5 *The Laplace transform of the modified Riemann-Liouville fractional derivative(Jumarie) of the function $f(t)$ is*

$$L\{f^{(\alpha)}(t)\}=s^{\alpha}l\{f(t)\}-s^{\alpha-1}f(0)-s^{\alpha-2}f'(0)-...-s^{\alpha-(n+1)}f^{(n)}(0),$$

for $n\leq\alpha<n+1,\ \ n\geq1$.

4 Formulation of the Casson model

Let us consider the transfer of heat of a Casson fluid on an unsteady boundary layer passing through an infinite oscillating plate in vertical direction, which is fixed at $y = 0$, the flow being restricted to $y > 0$, where y is the coordinate axis normal to the plate. Initially, for time $t = 0$, both plate and the fluid are at stational condition with the constant temperature T_∞. At time $t = 0^+$, the plate started to move in an oscillatory motion in its plane ($y = 0$) according to

$$U = VF(t)cos(\omega t)\vec{i};\ t > 0,$$

where $U = u(x,t)\vec{i}, V$ is the amplitude of the motion, $F(t)$ is the unit step function, $\vec{i}$ is the unit vector in the vertical direction of the flow and ω is the frequency of an oscillating plate. Concurrently, the transfer of heat from the plate to the fluid is directly proportional to the temperature T in the local surface. Let the rheological equation for an isotropic and incompressible Casson fluid, reported by Casson, is

$$\rho = \rho_0 + \lambda\beta,$$

correspondingly,

$$\rho_{ij} = \begin{cases} 2(\lambda_A + \frac{p_y}{\sqrt{2\pi}})\tau_{ij},\ \pi > \pi_e \\ 2(\lambda_A + \frac{p_y}{\sqrt{2\pi_e}})\tau_{ij},\ \pi < \pi_e. \end{cases}$$

Here ρ is the stress, ρ_0 is the Casson yield stress, λ is the dynamic viscosity, β is the shear rate, τ_{ij} is the $(i,j)^{th}$ component of the deformation rate, π is the product of the component of deformation rate with itself, that is $\pi = \tau_{ij}\tau_{ij}$, π_c is a critical value of this product based on the non-Newtonian case, λ_A is the plastic dynamic viscosity of the non-Newtonian fluid and p_y is the yield stress of fluid.

With these assumptions the heat transfer of a Casson fluid on an unsteady boundary layer is governed by momentum and energy equations, is given by

$$\begin{aligned} \mu\left[\frac{\partial U}{\partial t} + (U.\nabla)U\right] &= divT + \mu b, \\ \mu C_p\frac{\partial T}{\partial t} &= -\frac{\partial p}{\partial t} + k\nabla^2 T, \end{aligned}$$

where T is the Cauchy stress tensor, μ is the fluid density of the fluid, μb refers the body force, p is the pressure, C_p is the heat capacity at constant pressure and k is the thermal conductivity. Assume that the pressure is uniform across the boundary layer and by using the Boussinesq approximation, we get the set of partial differential equations as follows

$$\begin{aligned} \frac{\partial u}{\partial t} &= v(1 + \frac{1}{\beta})\frac{\partial^2 u}{\partial x^2} + s\gamma(T - T_\infty, \\ \mu C_p\frac{\partial T}{\partial t} &= k\frac{\partial^2 T}{\partial x^2}, \end{aligned}$$

with the initial and boundary conditions

$$
\begin{aligned}
u(x,0) &= 0, & T(x,0) = T_\infty, \quad \text{for all } x \geq 0,\\
u(0,t) &= F(t)V cos(\omega t), & \frac{\partial T}{\partial x}(0,t) = -h_s T(0,t), \quad t > 0,\\
u(\infty,t) &\to 0, & T(\infty,t) \to T_\infty, \quad t > 0,
\end{aligned}
$$

where u is the axial velocity, v is the kinematic viscosity, t is the time, β is the fluid parameter, s is the acceleration due to gravity, γ is the volumetric coefficient of thermal expansion and h_s is the heat transfer coefficient. The schematic diagram of the model(geometry of the problem) is presented in the Figure-1.

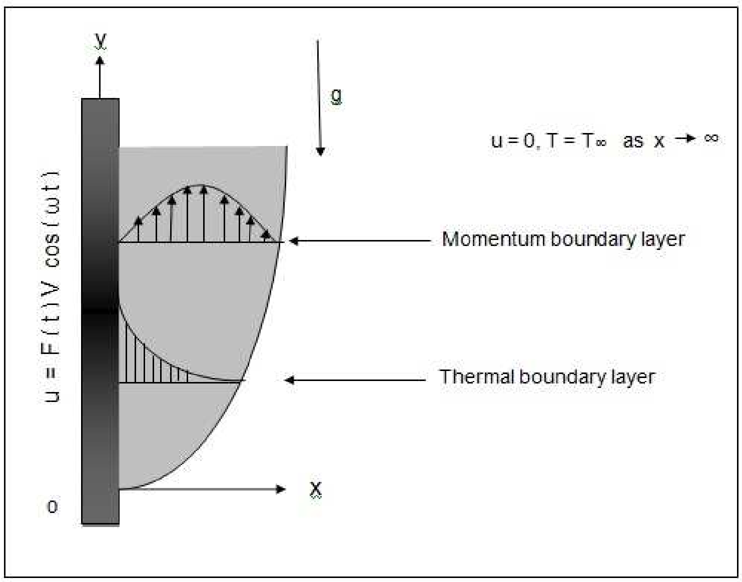

Figure 1: Schematic diagram of the model

We introduce the following non-dimensional quantities to shorten the above equations as,

$$
x^* = \frac{V}{v}x, \; t^* = \frac{V^2}{v}t, \; u^* = \frac{u}{V}, \; v = \frac{T - T_\infty}{T_\infty}, \; \omega^* = \frac{v}{V^2}\omega.
$$

Substituting the above quantities into the set of partial differential equations, we attain the following non-dimensional partial differential equations (*symbols are dropped out for simplicity)

$$
\frac{\partial u}{\partial t} = (1 + \frac{1}{\beta})\frac{\partial^2 u}{\partial x^2} + Gr \; v, \tag{4.1}
$$

$$
Pr\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}. \tag{4.2}
$$

Next the following equations describes the motion of the fractional generalized Casson fluid model

$$ {}^{J}D^{\alpha}u = (1+\frac{1}{\beta})\frac{\partial^2 u}{\partial x^2} + Gr\ v, \tag{4.3}$$

combined with the fractional diffusion heat equation

$$ {}^{J}D^{\alpha}v = \frac{1}{Pr}\frac{\partial^2 v}{\partial x^2}, \tag{4.4}$$

with the Goodman boundary conditions given by

$$ u(0,t) = v(0,t) = 1,\ u(\theta) = v(\kappa) = \frac{\partial u(\theta,t)}{\partial t} = \frac{\partial v(\kappa,t)}{\partial t},\ t \geq 0.$$

The parameters κ and θ denotes the penetration depths of the model and from the assumptions that for $x \geq \kappa$.

4.1 Solution of the formulated model

Here the algorithm of the solutions of the above model is described by adopting integral balance method: the heat balance integral and the double integral methods. The methods are considered as follow.
According to Goodman proposition, the approximate analytical solutions are provided as

$$ v(x,t) = \left[1-\frac{x}{\kappa}\right]^n, \tag{4.5}$$

where κ is the penetration depth as illustrated in the presentation of the model, and n is the exponent of the approximate solution, which will be determined by using the matching method. The second step consists of integrating the fractional differential equation between 0 to κ, taking into account the approximate solution (4.5) in the equation (4.4).

The last step consists of solving the obtained fractional differential equation after single integration between 0 to κ, which depends eventually on time. For the double integral methods, according to Goodman proposition, conserve the approximate analytical solution given in equation (4.5). The second step consists of applying the double integration on the fractional differential equation, the first integration between 0 to κ,and the second integration between x to κ by taking into account the approximate solution in the equation (4.4). The last step consists of solving the obtained fractional differential equation after the double integration.

To experiment with these methods, start the procedure of solutions by applying the heat balance integral method. Let the fractional diffusion equation defined by (4.4), and integrating between 0 to κ is

$$ \int_0^{\kappa} {}^{J}D^{\alpha}v dx = \int_0^{\kappa}\frac{1}{Pr}\frac{\partial^2 v}{\partial x^2}dx, $$
$$ {}^{J}D^{\alpha}\int_0^{\kappa} v dx = \frac{1}{Pr}\int_0^{\kappa}\frac{\partial^2 v}{\partial x^2}dx. $$

Using the approximate solution in the equation (4.4) and after some calculations, we obtain the following relationships

$$\begin{aligned}
{}^{J}D^{\alpha}\int_0^{\kappa}\left(1-\frac{x}{\kappa}\right)^n dx &= \frac{1}{Pr}\left[\frac{\partial v}{\partial x}\right]_0^{\kappa},\\
{}^{J}D^{\alpha}\left[-\frac{\delta}{n+1}\left(1-\frac{x}{\kappa}\right)^{n+1}\right]_0^{\kappa} &= \frac{1}{Pr}\left[\frac{-n}{\kappa}\left(1-\frac{x}{\kappa}\right)^{n-1}\right]_0^{\kappa},\\
{}^{J}D^{\alpha}\left[\frac{\kappa}{n+1}\right] &= \frac{1}{Pr}\frac{n}{\kappa},\\
\frac{1}{n+1}{}^{J}D^{\alpha}\kappa &= \frac{1}{Pr}\frac{n}{\kappa}.
\end{aligned}$$

Multiplying the above equation by $(n+1)\kappa$,

$$\begin{aligned}
\kappa^{J}D^{\alpha}\kappa &= \frac{n(n+1)}{Pr},\\
{}^{J}D^{\alpha}\kappa^2 &= \frac{2n(n+1)}{Pr},
\end{aligned} \tag{4.6}$$

Applying Laplace transform on both sides of the above expression,

$$\begin{aligned}
L\left[{}^{J}D^{\alpha}\kappa^2\right] &= \frac{2n(n+1)}{Pr}L[1],\\
s^{\alpha}L[\kappa^2] &= \frac{2n(n+1)}{sPr},\\
L[\kappa^2] &= \frac{2n(n+1)}{s^{\alpha+1}Pr}.
\end{aligned}$$

Now applying Inverse Laplace transform,

$$\kappa^2(t) = \frac{2n(n+1)t^{\alpha}}{Pr}.$$

Hence, the penetration depth is obtained such that the approximate analytical solution is given by

$$v(x,t) = \left[1-\frac{x}{\sqrt{\frac{2n(n+1)t^{\alpha}}{Pr}}}\right]^n. \tag{4.7}$$

To obtain the exponent value n, continue the resolution by proposing the approximate solution using the Double integral method. Applying the double integration between 0 to κ and between x to κ on the fractional diffusion equation,

$$\begin{aligned}
\int_0^{\kappa}\int_x^{\kappa}{}^{J}D^{\alpha}v\,dx\,dx &= \int_0^{\kappa}\int_x^{\kappa}\frac{1}{Pr}\frac{\partial^2 v}{\partial x^2}dx\,dx,\\
{}^{J}D^{\alpha}\int_0^{\kappa}\int_x^{\kappa}v\,dx\,dx &= \frac{1}{Pr}\int_0^{\kappa}\int_x^{\kappa}\frac{\partial^2 v}{\partial x^2}dx\,dx.
\end{aligned}$$

Using the approximation and calculation the integral between x to κ,

$$\begin{aligned}
{}^{J}D^{\alpha}\left[-\frac{\kappa^2}{(n+1)(n+2)}\left(1-\frac{x}{\kappa}\right)^{n+1}\right]_0^{\kappa} &= \frac{-1}{Pr}\int_0^{\kappa}\frac{\partial v}{\partial x}dx,\\
{}^{J}D^{\alpha}\left[\frac{\kappa^2}{(n+1)(n+2)}\right] &= \frac{1}{Pr}v(0,t),\\
\frac{1}{(n+1)(n+2)}\left[{}^{J}D^{\alpha}\kappa^2\right] &= \frac{1}{Pr}.
\end{aligned}$$

Applying Laplace transform of the Modified Riemann-Liouville fractional derivative,

$$\begin{aligned}
\frac{1}{(n+1)(n+2)}L\left[{}^{J}D^{\alpha}\kappa^2\right] &= \frac{1}{Pr}L[1],\\
s^{\alpha}L\left[\kappa^2\right] &= \frac{(n+1)(n+2)}{Pr}\frac{1}{s},\\
L\left[\kappa^2\right] &= \frac{(n+1)(n+2)}{s^{\alpha+1}Pr}.
\end{aligned}$$

Now applying Inverse Laplace transform,

$$\kappa^2(t) = \frac{(n+1)(n+2)}{Pr}t^{\alpha}.$$

Hence, the penetration depth is obtained such that the approximate analytical solution is given by

$$v(x,t) = \left[1-\frac{x}{\sqrt{\frac{(n+1)(n+2)t^{\alpha}}{Pr}}}\right]^n. \tag{4.8}$$

Hence from the above, it is observed that the exponent n appears again in the approximate solution with the Double integral method.

It is obvious to stipulate these approximations are the same and it is called the matching method. The value of the exponent is obtained as follows,

$$\begin{aligned}
\left[1-\frac{x}{\sqrt{\frac{2(n+1)t^{\alpha}}{Pr}}}\right]^n &= \left[1-\frac{x}{\sqrt{\frac{(n+1)(n+2)t^{\alpha}}{Pr}}}\right]^n,\\
n &= 2
\end{aligned}$$

Therefore, the approximate solution of the generalized fractional heat equation is obtained by considering the exponent $n = 2$, that is

$$v(x,t) = \left[1-\frac{x}{2\sqrt{\frac{3t^{\alpha}}{Pr}}}\right]^2$$

The second part of the resolution consists of applying the integral balance method and the double integral method in equation (4.3), that is, to solve the generalized fractional differential equation described by

$${}^{J}D^{\alpha}u = (1+\frac{1}{\beta})\frac{\partial^2 u}{\partial x^2} + Gr\left(1-\frac{x}{\kappa}\right)^n.$$

According to Goodman proposition, the approximate analytical solutions are described by

$$u(x,t) = \left(1 - \frac{x}{\theta}\right)^n,$$

Repeating the same procedure by applying the heat balance integral method. Taking the single integral between 0 to θ,

$$\begin{aligned}
\int_0^\theta {}^J D^\alpha u dx &= (1+\frac{1}{\beta})\int_0^\theta \frac{\partial^2 u}{\partial x^2}dx + Gr\int_0^\theta [1-\frac{x}{\kappa}]^n dx, \\
{}^J D^\alpha \int_0^\theta u dx &= (1+\frac{1}{\beta})\int_0^\theta \frac{\partial^2 u}{\partial x^2}dx + Gr\int_0^\theta [1-\frac{x}{\kappa}]^n dx, \\
{}^J D^\alpha \left[\frac{\theta}{n+1}\right] &= (1+\frac{1}{\beta})\frac{n}{\theta} + \frac{Gr\kappa}{n+1} - \frac{Gr\theta}{n+1}\left(1-\frac{\theta}{\kappa}\right).
\end{aligned}$$

Rearranging the last term and multiplying by the parameter $(n+1)\theta$,

$${}^J D^\alpha \theta^2 = 2(1+\frac{1}{\beta})n(n+1) + 2Gr\theta^2.$$

Applying the Laplace transform of the Modified Riemann-Liouville fractional derivative,

$$\begin{aligned}
(1-2Gr)s^\alpha L[\theta^2] &= \frac{2n(n+1)}{s}(1+\frac{1}{\beta}), \\
L[\theta^2] &= \frac{2n(n+1)}{s^{\alpha+1}}(1+\frac{1}{\beta}).
\end{aligned}$$

By applying Inverse Laplace transform,

$$\theta^2(t) = \frac{2n(n+1)}{1-2Gr}(1+\frac{1}{\beta})t^\alpha.$$

Thus, the approximate analytical solution of the equation (4.3) is given by

$$u(x,t) = \left[1 - \frac{x}{\sqrt{\frac{2(1+\frac{1}{\beta})n(n+1)t^\alpha}{[1-2Gr]}}}\right]^n. \tag{4.9}$$

Next applying the double integral between 0 to θ and between x to θ on the fractional diffusion equation (1),

$$\begin{aligned}
\int_0^\theta \int_x^\theta {}^J D^\alpha u dx dx &= (1+\frac{1}{\beta})\int_0^\theta \int_x^\theta \frac{\partial^2 u}{\partial x^2}dx dx + Gr\int_0^\theta \int_x^\theta [1-\frac{x}{\kappa}]^n dx dx, \\
{}^J D^\alpha \int_0^\theta \int_x^\theta u dx dx &= (1+\frac{1}{\beta})\int_0^\theta \int_x^\theta \frac{\partial^2 u}{\partial x^2}dx dx + Gr\int_0^\theta \int_x^\theta [1-\frac{x}{\kappa}]^n dx dx, \\
{}^J D^\alpha \left[-\frac{\theta^2}{(n+1)(n+2)}\left(1-\frac{x}{\theta}\right)^{n+1}\right]_0^\theta &= -(1+\frac{1}{\beta})\int_0^\theta \frac{\partial u}{\partial x}dx + \frac{Gr\theta^2}{(n+1)(n+2)}, \\
{}^J D^\alpha \left[\frac{\theta^2}{(n+1)(n+2)}\right] &= (1+\frac{1}{\beta}) + \frac{Gr\theta^2}{(n+1)(n+2)}.
\end{aligned}$$

By using Laplace transform on both sides and arranging the like terms,

$$\begin{aligned} s^\alpha L[\theta^2(s)] &= (1+\frac{1}{\beta})\frac{(n+1)(n+2)}{s} + \frac{Gr.L[\theta^2(s)]}{(n+1)(n+2)}, \\ L[\theta^2(s)] &= (1+\frac{1}{\beta})\frac{(n+1)(n+2)}{[1-2Gr]s^{\alpha+1}}. \end{aligned}$$

Applying the Inverse Laplace transform,

$$\theta^2(t) = (1+\frac{1}{\beta})\frac{(n+1)(n+2)}{[1-2Gr]}t^\alpha.$$

Thus, the approximate analytical solution is obtained by

$$u(x,t) = \left[1 - \frac{x}{\sqrt{\frac{(1+\frac{1}{\beta})(n+1)(n+2)t^\alpha}{[1-2Gr]}}}\right]^n.$$

From the above, it is observed that the exponent n appears in the approximate of the penetration depth with the double integral method also. So to find the value of the exponent,

$$\begin{aligned} \frac{2(1+\frac{1}{\beta})n(n+1)t^\alpha}{[1-2Gr]} &= \frac{(1+\frac{1}{\beta})(n+1)(n+2)t^\alpha}{[1-2Gr]}. \\ n &= 2. \end{aligned}$$

Hence, the approximate solution of the generalized fractional heat equation by taking the exponent value $n = 2$ is

$$u(x,t) = \left[1 - \frac{x}{2\sqrt{\frac{3(1+\frac{1}{\beta})t^\alpha}{[1-2Gr]}}}\right]^2$$

5 Graphical Interpretations and Discussion

This is the main section of the study that analyzes the impact of the modified Riemann-Liouville fractional derivative into the diffusion process, especially into the velocity and temperature. The effects of the fractional order into the velocity of the Casson fluid in x-direction. The Physical behavior of the embedded parameters such as Grashold number Gr, Prandtl number Pr, fractional order α and the Casson diffusion coefficient μ are plotted in graphs (Figure 2-6) and discussed in detail.

The temperature profiles for different values of an order of modified Riemann-Liouville fractional derivative, α are shown in the Figure-2. Let us fix the Prandtl number Pr$= 0.25$ and the time $t = 0.3$. We know that the fractional derivative α varies from 0 to 1. In Figure-2, it is observed that the temperature trajectories decrease according to the increase of the direction x. Further, it is noticed that when the parameter α increases then all the velocities decrease in the following direction as indicated by an arrow. Also, the fractional

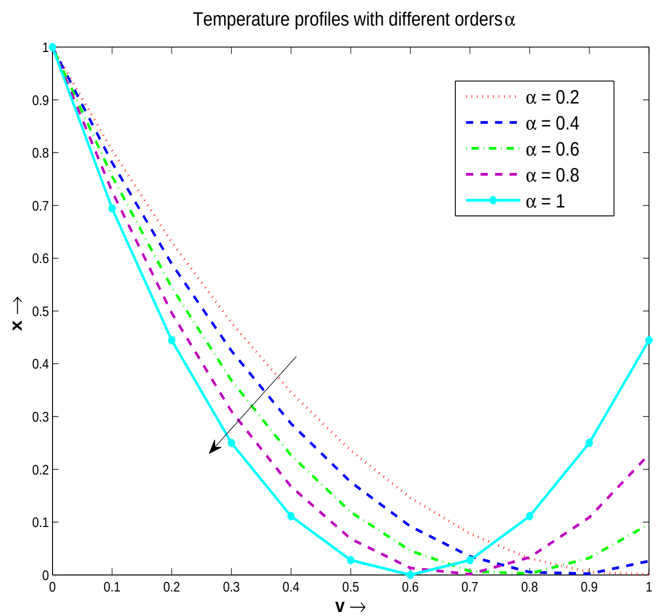

Figure 2: Temperature profiles with differing values of α, when $Pr = 0.25, t = 0.3$

order α does not have any impact as the Casson fluid moves apart from the boundary region and we conclude that α has an acceleration effect.

The temperature profiles are shown in Figure-3 for different values of Prandtl number Pr. Let us fix the fractional order $\alpha = 0.95$ and the time $t = 0.3$ into the temperature diffusion. From Figure-3, it is noted that the temperature decreases more quickly with increasing of the Prandtl number.

From this, one can observed that the fluids with maximum Prandtl number have high viscosity and small thermal conductivity, which makes the fluid thick and therefore causes a decrease in temperature of the Casson fluid. Finally, it is concluded that the increase of the Prandtl number Pr accelerates the decrease of the temperature to the value zero.

Next by continuing to analyze the velocity profiles for different values of the fractional order α, the Grashold number Gr and the diffusion coefficient μ are shown below. In Figure-4, as in the previous analysis, we fix the value of the time $t = 0.3$, the Grashold number $Gr = 0.25$ and the diffusion coefficient $\mu = 5$ and plotted the velocity curve for different fractional order α varies from 0 to 1. Here, it is noted that when the order α increases, then all the velocities are decreases in the direction indicated by an arrow shown in the Figure-4. Also the fractional order α also has an acceleration effect into the diffusion process.

Now we assume the velocity of the Casson fluid. The diffusion coeffecient $\mu = 5$ is fixed and the order $\alpha = 0.95$. From Figure-5 , we note that when the Grashold number Gr

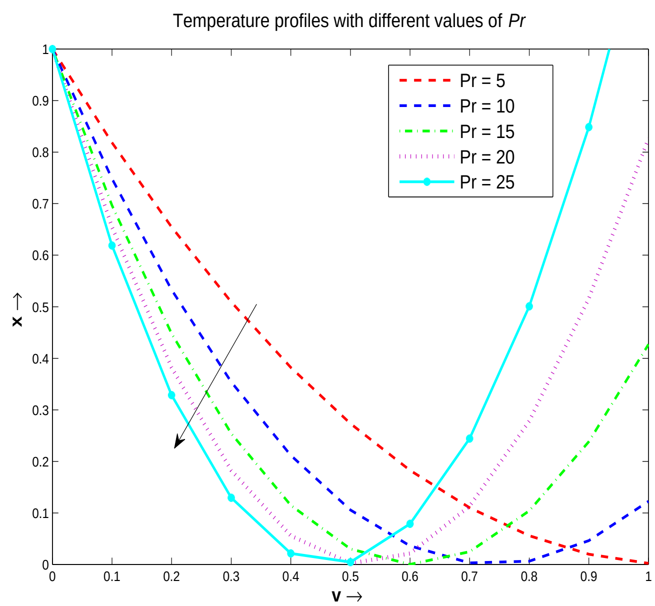

Figure 3: Temperature profiles with differing values of Pr, when $\alpha = 0.95, t = 0.3$

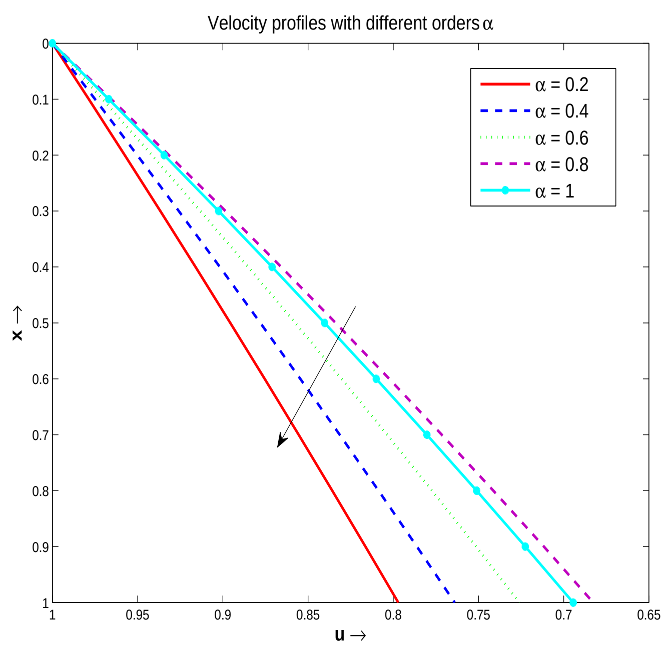

Figure 4: Velocity profiles with differing values of α, when $Gr = 0.25, \mu = 5, t = 0.3$

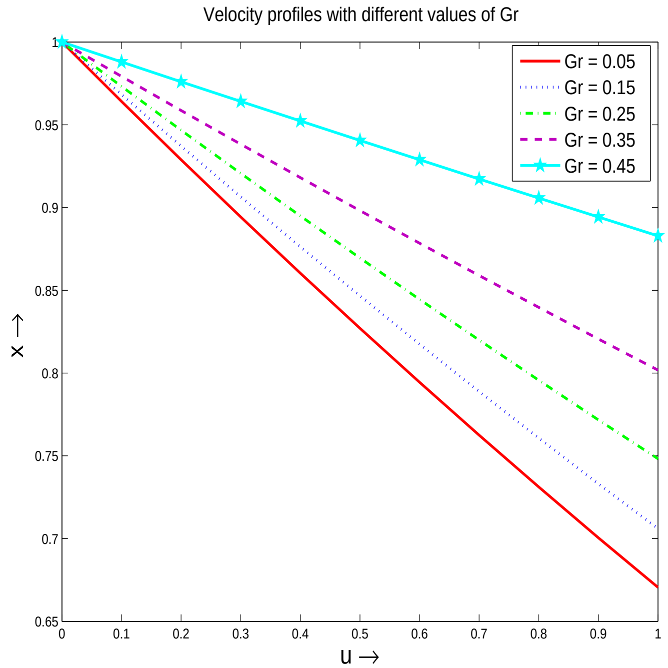

Figure 5: Velocity profiles with differing values of Gr, when $\alpha = 0.95, \mu = 5, t = 0.3$

increases, the velocities increase to one another. Grashold number also has an acceleration effect that is it accelerate the indensity of the velocity to a value which is different to zero, indicated by an arrow as shown in the figure. Also observed that the velocity curves of the fluid attains its maximum near the plate and reaches to zero as $x \to 1^+(\infty)$. In addition to this, it is found that the response of α and Pr on the velocity profiles are indistinguishable.

The Casson diffusion co-efficient μ also induces the same behavior, when we fix the Grashold number $Gr = 0.25$, the fractional order $\alpha = 0.5$. From Figure - 6, it is observed that when the coefficient increases, then all the velocities decrease to one another and the general behavior of the velocities are also decreases. The acceleration effects that the intensity of the velocity approaches to zero is indicated by an arrow in the direction shown in the figure.

6 Conclusion

In this study, an analytical solution for the unsteady boundary layer flow of a generalized fractional casson fluid model passing through an infinite oscillating plate in vertical direction with Newtonian heating described by modified Riemann-Liouville fractional derivative are obtained by using Integral balance method and Laplace transform technique with the help of Goodman boundary conditions. In order to support the main result, graphical interpretations are implemented. A remarkable findings of this study is that the separation of the flow can be governed by increasing the value of the Prandtl number and also the value

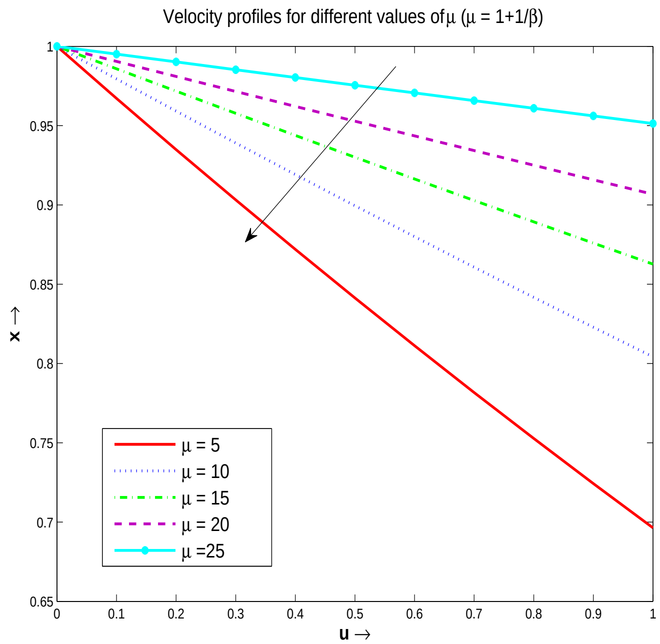

Figure 6: Velocity profiles with differing values of μ, when $\alpha = 0.5, Gr = 0.25, t = 0.3$

of the Casson fluid parameter. The methods employed in this study will create a new path towards approximating the solutions of the Casson fluid models.

The following points have been extracted from this study

- Velocity $u(x,t)$ is inversely proportional to x, μ and directly proportional to Gr.
- Temperature $v(x,t)$is inversely proportional to α and Pr.

The numerical results for velocity and temperature are computed in Table-1.

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x	μ	t	α	Gr	Pr	$u(x,t)$	$v(x,t)$
0	5	0.3	0.6	0.25	0.25	1	1
0.25	5	0.3	0.6	0.3	0.3	0.9423	0.8898
0.25	10	0.4	0.6	0.25	0.25	0.9580	0.9073
0.25	15	0.3	1	0.25	0.25	0.9525	0.8726
0.5	5	0.3	0.6	0.3	0.3	0.8863	0.7860
0.5	10	0.4	0.6	0.25	0.25	0.9168	0.8190
0.5	15	0.3	1	0.25	0.25	0.9061	0.7538
0.75	5	0.3	0.6	0.3	0.3	0.8320	0.6886
0.75	10	0.4	0.6	0.25	0.25	0.8766	0.7353
0.75	15	0.3	1	0.25	0.25	0.8609	0.6437
1	5	0.3	0.6	0.3	0.3	0.7794	0.5979
1	10	0.4	0.6	0.25	0.25	0.8373	0.6561
1	15	0.3	1	0.25	0.25	0.8168	0.5424
1.25	5	0.3	0.6	0.3	0.3	0.7285	0.5132
1.25	10	0.4	0.6	0.25	0.25	0.7989	0.5814
1.25	15	0.3	1	0.25	0.25	0.7739	0.4497
1.5	5	0.3	0.6	0.3	0.3	0.6794	0.4351
1.5	10	0.4	0.6	0.25	0.25	0.7613	0.5112
1.5	15	0.3	1	0.25	0.25	0.7322	0.3657
1.75	5	0.3	0.6	0.3	0.3	0.6319	0.3635
1.75	10	0.4	0.6	0.25	0.25	0.7247	0.4455
1.75	15	0.3	1	0.25	0.25	0.6916	0.2903
2	5	0.3	0.6	0.3	0.3	0.5862	0.2983
2	10	0.4	0.6	0.25	0.25	0.6890	0.3844
2	15	0.3	1	0.25	0.25	0.6521	0.2237

Table 1: Numerical results for velocity $u(x,t)$ and temperature $v(x,t)$

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