UNIQUENESS OF SOLUTIONS, STABILITY AND SIMULATIONS FOR A DIFFERENTIAL PROBLEM INVOLVING CONVERGENT SERIES AND TIME VARIABLE SINGULARITIES

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Abstract. We focus on a new type of nonlinear integro-differential equations with nonlocal integral conditions. The considered problem has one nonlinearity with time variable singularity. It involves also some convergent series combined to Riemann-Liouville integrals. We prove a uniqueness of solutions for the proposed problem, then, we provide some examples to illustrate this result. Also, we discuss the Ulam-Hyers stability for the problem. Some numerical simulations, using Rung Kutta method, are discussed too. At the end, a conclusion follows.

1. Introduction

The fractional differential equations theory is one of the theories of modern mathematics that has received a great deal of attention in the last two decades. It is used in modeling many phenomena in mechanics, chemistry, biology, economics, etc. We refer the reader to [2, 3, 4, 7, 8, 14] for more information and also for some important applications. Moreover, the differential equations, with time or space singularities, are of great interest since several physical situations are modelled by problems of this kind, (for example, problems in gas and fluid dynamics), see [5, 6, 17]. In this singular field theory, many authors have paid a great attention to the questions of the existence and uniqueness of solutions to this type of equations. For more details, we refer the reader to [10, 12, 13]. The reader can also point out that stability of solutions of such equations is useful in solving many problems in economics, mechanics, and also in control theory, see [13, 19, 20] and the reference therein.
Before introducing our problem, we need to cite some other results that have motivated our aim. We begin by [1], where the authors have studied, for the first time, the existence and uniqueness of solutions for the following non singular system involving series:

$$\begin{cases} D^{\alpha}u(t)=f_1(t,u(t),v(t))+\displaystyle\sum_{i=1}^{\infty}\int_0^t\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\varphi_i(s)g_i(s,u(s),v(s))ds,\, t\in[0,1],\\ D^{\beta}v(t)=f_2(t,u(t),v(t))+\displaystyle\sum_{i=1}^{\infty}\int_0^t\frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}\phi_i(s)h_i(s,u(s),v(s))ds,\, t\in[0,1],\\ \displaystyle\sum_{k=0}^{n-2}(|u^{(k)}(0)|+|v^{(k)}(0)|)=0,\\ u^{(n-1)}(0)=\gamma I^{p}u(\eta),\eta\in[0,1],\\ v^{(n-1)}(0)=\delta I^{p}v(\zeta),\zeta\in[0,1].\end{cases}$$

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Then, based on the above paper, the authors in [18] have studied the following second non singular fractional differential problem:

$$
\begin{cases}
D^{\alpha_1} u(t) = \sum_{i=1}^{l} f_i(t, u(t), v(t), D^{\gamma_1} u(t), D^{\gamma_1} v(t)) \\
+ \sum_{j=1}^{\infty} \int_0^t \frac{(t-s)^{\delta_j - 1}}{\Gamma(\delta_j)} \varphi_i(s) g_i(s, u(s), v(s), D^{\gamma_2} u(s), D^{\gamma_2} v(s)) ds, t \in [0,1], \\
D^{\alpha_2} v(t) = \sum_{i=1}^{l} k_i(t, u(t), v(t), D^{\gamma_2} u(t), D^{\gamma_2} v(t)) \\
+ \sum_{j=1}^{\infty} \int_0^t \frac{(t-s)^{\theta_j - 1}}{\Gamma(\theta_j)} \phi_i(s) h_i(s, u(s), v(s), D^{\gamma_2} u(s), D^{\gamma_2} v(s)) ds, t \in [0,1], \\
\\
u(0) = a_0, v(0) = b_0, \\
u^{(j)}(0) = v^{(j)}(0) = 0, j = 1, 2, ..., n-2, \\
u^{(n-1)}(0) = J^p u(\tau), p > 0, \tau \in]0,1[, \\
v^{(n-1)}(0) = J^q v(\rho), q > 0, \rho \in]0,1[.
\end{cases}
$$

For the singular case without series, we can also cite the papers [11, 12], where the authors have studied the questions of existence of solutions as well as the stability for the problem:

$$
\begin{cases}
D^{\beta}(D^{\alpha} + \frac{k}{t^{\lambda}}) y(t) + \Delta_1 f(t, y(t), D^{\delta} y(t)) + \Delta_2 g(t, y(t), I^{\rho} y(t)) + h(t, y(t)) \\
= l(t), \ t \in]0,1[, \\
\\
y(0) = 0, \\
y(1) = b \int_0^{\eta} y(s) ds, \ 0 < \eta < 1, \\
I^q y(u) = y(1), \ 0 < u < 1, \\
k > 0, 0 < \lambda \leq 1, \ 1 \leq \beta \leq 2, \ 0 \leq \alpha, \ \delta \leq 1,
\end{cases}
$$

where, $\Delta_1 > 0$, $\Delta_2 > 0$, $J := [0,1]$, the two fractional derivative of the problem are in the sense of Caputo, I^{ρ} is the Riemann-Liouville integral and f, g, h, l are some given functions.
Motivated by both the above two series-works and by the applications of singular differential equations in fluid dynamics, in this paper, we study the following problem:

$$
(1.1) \qquad
\begin{cases}
D^{\alpha} u(t) + \lambda f(u(t), u''(t)) = \delta g(t, u(t), D^{\gamma} u(t)) + \sum_{i=1}^{\infty} \nu_i \Phi_i(t) I^{\alpha} h_i(t, u(t)), t \in (0,1], \\
u(0) + u(1) = \int_0^{\eta} \kappa_1 u(s) ds, \ \ 0 < \eta < 1, \\
u'(0) + u'(1) = \int_0^{\theta} \kappa_2 u(s) ds, \ \ 0 < \theta < 1, \\
u''(0) + u''(1) = \int_0^{\tau} \kappa_3 u(s) ds, \ \ 0 < \tau < 1, \\
\\
2 < \alpha \leq 3, \ \ 0 < \gamma < 1, \ \ \lambda, \delta, \nu_i \in \mathbb{R},
\end{cases}
$$

where we note that $J := [0,1]$, the functions f and h_i will be specified later, g is singular at $t = 0$, the operators D^{α} and D^{γ} are the derivatives in the sense of Caputo.
To the best of our knowledge, this is the first time in the literature where singular differential equations, involving fractional calculus and convergent series on Riemann-Liouville integrals and other terms, are investigated. So, in general, our aim is to present a first contribution in this direction and try to fill this gap. Especially, we study the question of existence and uniqueness of solutions by using

both fixed point theory and integral inequalities, then we pass to the investigate the question of stability of solutions in the sense of Ulam-Hyers where the integral inequalities and estimates will allow us to prove the results. Our results will be concretized by some illustrated examples. Then, thanks to some numerical techniques that allow us to approximate the Caputo derivatives, (see the two papers [9, 15]), and by using Rung Kutta method, we present a numerical study with some simulations in order to present to the reader more comprehension on the proposed examples.

2. Caputo Approach Preliminaries

We need to introduce the Caputo derivatives. For more details, we refer to the reference [16].

Definition 1. *Let $\alpha > 0$ and $f : J \longmapsto \mathbb{R}$ be a continuous function. The Riemann-Liouville integral is defined by:*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Definition 2. *For any $f \in C^n(J,\mathbb{R})$ and $n-1 < \alpha \leq n$, the Caputo derivative is defined by:*

$$\begin{aligned} D^{\alpha} f(t) &= I^{n-\alpha} \frac{d^n}{dt^n}(f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds. \end{aligned}$$

To study (1.1), we need the following two results [16]:

Lemma 1. *Let $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$. Then, the general solution of $D^{\alpha} y(t) = 0; t \in J$ is:*

$$y(t) = \sum_{i=0}^{n-1} c_i t^i,$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, .., n-1$.

Lemma 2. *If $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$, then, we have*

$$I^{\alpha} D^{\alpha} y(t) = y(t) + \sum_{i=0}^{n-1} c_i t^i,$$

and $c_i \in \mathbb{R}$, $i = 0, 1, 2, .., n-1$.

Now, we present to the reader the proof of the integral solution of our problem.

Lemma 3. *Let the functions G be in the space $C(]0,1])$ and $(H_i)_{i=1,\ldots,r}, r \in \mathbb{N}^*$, Φ_i be in $C(J)$, such that $\sum_{i=1}^{\infty} \|\nu_i \Phi_i H_i\|_{\infty}$ is finite. Then, the differential problem*

$$\begin{cases} D^{\alpha} u(t) = G(t) + \sum_{i=1}^{\infty} \nu_i \Phi_i(t) H_i(t), t \in (0,1], \\ u(0) + u(1) = \int_0^{\eta} \kappa_1 u(s) ds, \quad 0 < \eta < 1, \\ u'(0) + u'(1) = \int_0^{\theta} \kappa_2 u(s) ds, \quad 0 < \theta < 1, \\ u''(0) + u''(1) = \int_0^{\tau} \kappa_3 u(s) ds, \quad 0 < \tau < 1, \\ \\ 2 < \alpha \leq 3, \quad \nu_i \in \mathbb{R}, \end{cases}$$

admits as integral representation the equation:

$$u(t) \qquad = I^{\alpha} G(t) + \sum_{i=1}^{\infty} \nu_i I^{\alpha}\Big(\Phi_i(t) I^{\alpha} H_i(t)\Big) + \Big[\frac{\Lambda_1 t^2 + \psi_1 t + \Delta_1}{\varphi}\Big]\Big[\kappa_3 \int_0^{\eta} I^{\alpha} G(s) ds$$

$$
\begin{aligned}
&+\sum_{i=1}^{\infty}\kappa_3\nu_i\int_0^{\eta} I^{\alpha}\Big(\Phi_i(s)I^{\alpha}H_i(s)\Big)ds-\frac{1}{\Gamma(\alpha-2)}\int_0^1(1-s)^{\alpha-3}G(s)ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-2)}\\
&\quad\times\int_0^1(1-s)^{\alpha-3}\Big(\Phi_i(s)I^{\alpha}H_i(s)\Big)ds\Bigg]+\left[\frac{\Lambda_2t^2+\psi_2t+\Delta_2}{\varphi}\right]\Bigg[\kappa_2\int_0^{\theta}I^{\alpha}G(s)ds\\
&+\sum_{i=1}^{\infty}\kappa_2\nu_i\int_0^{\theta} I^{\alpha}\Big(\Phi_i(s)I^{\alpha}H_i(s)\Big)ds-\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}G(s)ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-1)}\\
&\quad\times\int_0^1(1-s)^{\alpha-2}\Big(\Phi_i(s)I^{\alpha}H_i(s)\Big)ds\Bigg]+\left[\frac{\Lambda_3t^2+\psi_3t+\Delta_3}{\varphi}\right]\Bigg[\kappa_1\int_0^{\xi}I^{\alpha}G(s)ds\\
&+\sum_{i=1}^{\infty}\kappa_1\nu_i\int_0^{\xi} I^{\alpha}\Big(\Phi_i(s)I^{\alpha}H_i(s)\Big)ds-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}G(s)ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha)}\\
&\quad\times\int_0^1(1-s)^{\alpha-1}\Big(\Phi_i(s)I^{\alpha}H_i(s)\Big)ds\Bigg].
\end{aligned}
\tag{2.1}
$$

where, it is to note that:

$$
\begin{aligned}
\varphi &= F_1(E_2-2)(D_3-1)+E_1(F_3-2)(D_2-2)+(D_1-4)F_2(E_3-1)-F_1(E_3-1)(D_2-2)\\
&\quad-(F_3-2)(E_2-2)(D_1-4)-E_1F_2(D_3-1),
\end{aligned}
$$

$$\Lambda_1 = (F_3-2)(E_2-2)-F_2(E_3-1),$$

$$\Lambda_2 = F_1(E_3-1)-E_1(F_3-2),$$

$$\Lambda_3 = E_1F_2-F_1(E_2-2),$$

$$\psi_1 = F_2(D_3-1)-(F_3-2)(D_2-2),$$

$$\psi_2 = (F_3-2)(D_1-4)-F_1(D_3-1),$$

$$\psi_3 = F_1(D_2-2)-F_2(D_1-4),$$

$$\Delta_1 = (E_3-1)(D_2-2)-(E_2-2)(D_3-1),$$

$$\Delta_2 = E_1(D_3-1)-(E_3-1)(D_1-4),$$

$$\Delta_3 = (E_2-2)(D_1-4)-E_1(D_2-2),$$

$$
\begin{aligned}
D_1 &= \frac{\kappa_3\eta^3}{3}, E_1=\frac{\kappa_3\eta^2}{2}, F_1=\kappa_3\eta,\\
D_2 &= \frac{\kappa_2\theta^3}{3}, E_2=\frac{\kappa_2\theta^2}{2}, F_2=\kappa_3\theta,\\
D_3 &= \frac{\kappa_1\xi^3}{3}, E_3=\frac{\kappa_1\xi^2}{2}, F_3=\kappa_1\xi,
\end{aligned}
$$

and $\varphi\neq 0$.

Proof: Thanks to Lemma 2, we observe that

$$u(t) = I^{\alpha}G(t) + \sum_{i=1}^{\infty}\nu_i I^{\alpha}\Big(\Phi_i(t)I^{\alpha}H_i(t)\Big) + c_2t^2 + c_1t + c_0,$$
$$u'(t) = I^{\alpha-1}G(t) + \sum_{i=1}^{\infty}\nu_i I^{\alpha-1}\Big(\Phi_i(t)I^{\alpha}H_i(t)\Big) + 2c_2t + c_1, \tag{2.2}$$
$$u''(t) = I^{\alpha-2}G(t) + \sum_{i=1}^{\infty}\nu_i I^{\alpha-2}\Big(\Phi_i(t)I^{\alpha}H_i(t)\Big) + 2c_2,$$

By considering the conditions

$$u(0)+u(1) = \int_0^{\eta}\kappa_1 u(s)ds, \quad 0<\eta<1$$
$$u'(0)+u'(1) = \int_0^{\theta}\kappa_2 u(s)ds, \quad 0<\theta<1,$$
$$u''(0)+u''(1) = \int_0^{\tau}\kappa_3 u(s)ds, \quad 0<\tau<1$$

and thanks to Cramer rule, we achieve the proof.

In what follows, we use fixed point theory to study the above problem. First, it is important to introduce the space:

$$X := \{x\in C(J,\mathbb{R}), x''\in C(J,\mathbb{R}), D^{\gamma}x\in C(J,\mathbb{R})\},$$

and the norm:

$$\|x\|_X = \|x\|_{\infty} + \|x''\|_{\infty} + \|D^{\gamma}x\|_{\infty}.$$

Then, we shall consider the nonlinear operator $H: X\to X$ defined by by:

$$\begin{aligned}
Hu(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\sigma g(s,u(s),D^{\gamma}u(s)) - \lambda f(u(s),u''(s))]ds \\
&+ \sum_{i=1}^{\infty}\nu_i\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}h_i(\tau,u(\tau))d\tau\Big)ds \\
&+\Big[\frac{\Lambda_1t^2+\psi_1t+\Delta_1}{\varphi}\Big]\Big[\kappa_3\int_0^{\eta}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\sigma g(\tau,u(\tau),D^{\gamma}u(\tau)) - \lambda f(u(\tau),u''(\tau))]d\tau ds \\
&+ \sum_{i=1}^{\infty}\kappa_3\nu_i\int_0^{\eta}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}[\sigma g(s,u(s),D^{\gamma}u(s)) - \lambda f(u(s),u''(s))]ds - \sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-2)} \\
&\times\int_0^1 (1-s)^{\alpha-3}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big] + \Big[\frac{\Lambda_2t^2+\psi_2t+\Delta_2}{\varphi}\Big] \\
&\Big[\kappa_2\int_0^{\theta}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\sigma g(\tau,u(\tau),D^{\gamma}u(\tau)) - \lambda f(u(\tau),u''(\tau))]d\tau ds \\
&+ \sum_{i=1}^{\infty}\kappa_2\nu_i\int_0^{\theta}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}[\sigma g(s,u(s),D^{\gamma}u(s)) - \lambda f(u(s),u''(s))]ds - \sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-1)} \\
&\times\int_0^1 (1-s)^{\alpha-2}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big] + \Big[\frac{\Lambda_3t^2+\psi_3t+\Delta_3}{\varphi}\Big]
\end{aligned}$$

$$\begin{aligned}
&\Big[\kappa_1 \int_0^{\xi} \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1}[\sigma g(\tau,u(\tau),D^{\gamma}u(\tau)) - \lambda f(u(\tau),u''(\tau))]d\tau ds \\
&+\sum_{i=1}^{\infty} \kappa_1 \nu_i \int_0^{\xi} \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}[\sigma g(s,u(s),D^{\gamma}u(s)) - \lambda f(u(s),u''(s))]ds - \sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha)} \\
&\times \int_0^1 (1-s)^{\alpha-1}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big]. \qquad (2.3)
\end{aligned}$$

At the end of this section, it is important to note that in this paper, we will be concerned with singular differential equations, fixed point theory nd integral inequalities to prove our main results.

3. Unique Solutions

We consider the following sufficient hypotheses:

$(Q1)$: The functions f is defined on $\mathbb{R}^2$, g is defined on $(0,1]\times\mathbb{R}^2$ and h_i are defined on $J\times\mathbb{R}$; all these functions are supposed continuous continuous.

$(Q2)$: There exist positive continuous functions $\varpi_1(t), \varpi_2(t), \iota_1(t), \iota_2(t), \varsigma(t)$, such that for any $t \in J$, $u_i, v_i \in \mathbb{R}$,

$$\begin{aligned}
|f(u_1,u_2) - f(v_1,v_2)| &\leq \varpi_1(t)\frac{|u_1-v_1|}{1+|u_1+u_2|} + \varpi_2(t)\frac{|u_2-v_2|}{1+|v_1+v_2|}, \\
|g(t,u_1,u_2) - g(t,v_1,v_2)| &\leq \iota_1(t)\sin(u_1-v_1) + \iota_2(t)\frac{|u_2-v_2|}{1+|u_2v_2|}.
\end{aligned}$$

and for any integer i and any $t \in J$, $u,v \in \mathbb{R}$

$$|h_i(t,u) - h_i(t,v)| \leq \varsigma(t)\frac{|u-v|}{(1+t^2)(|u|+|v|)}.$$

We take the expressions:

$$\begin{gathered}
N = Max(\sup_{t\in J}|\varpi_1(t)|, \sup_{t\in J}|\varpi_2(t)|), \\
M = Max(\sup_{t\in J}|\iota_1(t)|, \sup_{t\in J}|\iota_2(t)|), \\
O = \sup_{t\in J}|\varsigma(t)|.
\end{gathered}$$

$(Q3)$: Suppose that Φ_i are defined on J, continuous and $\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_{\infty} < +\infty$.

Also we consider:

$$
\begin{aligned}
\Upsilon_1 &= \left[\frac{M|\delta|+N|\lambda|}{\Gamma(\alpha+1)} + \frac{O}{\Gamma(2\alpha+1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right] + \left[\frac{|\Lambda_1|+|\psi_1|+|\Delta_1|}{|\varphi|}\right]\times\left[\left(M|\delta|+N|\lambda|\right)\right.\\
&\times\left(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha-1)}\right) + \left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left.\left(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha-1)}\right)\right]\\
&+\left[\frac{|\Lambda_2|+|\psi_2|+|\Delta_2|}{|\varphi|}\right]\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha)}\right) + \left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\right.\\
&\times\left.\left(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha)}\right)\right] + \left[\frac{|\Lambda_3|+|\psi_3|+|\Delta_3|}{|\varphi|}\right]\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\right)\right.\\
&+\left.\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha+1)}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_2 &= \left[\frac{M|\delta|+N|\lambda|}{\Gamma(\alpha-\gamma+1)} + \frac{O}{\Gamma(2\alpha-\gamma+1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right] + \frac{1}{|\varphi|}\left[\frac{2|\Lambda_1|}{\Gamma(3-\gamma)} + \frac{|\psi_1|}{\Gamma(2-\gamma)}\right]\\
&\times\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha-1)}\right) + \left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)}\right.\right.\\
&+\left.\left.\frac{1}{\Gamma(2\alpha-1)}\right)\right] + \frac{1}{|\varphi|}\left[\frac{2|\Lambda_2|}{\Gamma(3-\gamma)} + \frac{|\psi_2|}{\Gamma(2-\gamma)}\right]\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha)}\right)\right.\\
&+\left.\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha)}\right)\right] + \frac{1}{|\varphi|}\left[\frac{2|\Lambda_3|}{\Gamma(3-\gamma)} + \frac{|\psi_3|}{\Gamma(2-\gamma)}\right]\\
&\times\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\right) + \left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\right.\\
&\times\left.\left(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha+1)}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_3 &= \left[\frac{M|\delta|+N|\lambda|}{\Gamma(\alpha-1)} + \frac{O}{\Gamma(2\alpha-1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right] + \frac{2|\Lambda_1|}{|\varphi|}\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)}\right.\right.\\
&+\left.\frac{1}{\Gamma(\alpha-1)}\right) + \left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left.\left(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha-1)}\right)\right] + \frac{2|\Lambda_2|}{|\varphi|}\\
&\times\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha)}\right) + \left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha)}\right)\right]\\
&+\frac{2|\Lambda_3|}{|\varphi|}\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\right) + \left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\right.\\
&\times\left.\left(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha+1)}\right)\right].
\end{aligned}
$$

Theorem 1. *Assume that both the three hypotheses* $(Q_1), (Q_2), (Q_3)$ *and the condition* $\Upsilon < 1; \Upsilon = \Upsilon_1 + \Upsilon_2 + \Upsilon_3$ *are satisfied. Then, the problem* (1.1) *has a unique solution.*

Proof: We begin this proof by showing that H satisfies the Banach contraction principle. For $(u,v)\in X^2$, we can write

$$\begin{aligned}
&\|Hu-Hv\|_\infty\\
&\leq\left[\frac{M|\delta|+N|\lambda|}{\Gamma(\alpha+1)}+\frac{O}{\Gamma(2\alpha+1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right]\|u-v\|_X+\left[\frac{|\Lambda_1|+|\psi_1|+|\Delta_1|}{|\varphi|}\right]\\
&\times\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha-1)}\right)+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\right.\\
&\times\left.\left(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha-1)}\right)\right]\|u-v\|_X+\left[\frac{|\Lambda_2|+|\psi_2|+|\Delta_2|}{|\varphi|}\right]\left[\left(M|\delta|+N|\lambda|\right)\right.\\
&\times\left(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha)}\right)+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left.\left(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha)}\right)\right]\|u-v\|_X\\
&+\left[\frac{|\Lambda_3|+|\psi_3|+|\Delta_3|}{|\varphi|}\right]\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}\right)\right.\\
&+\left.\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha+1)}\right)\right]\|u-v\|_X.
\end{aligned}\tag{3.1}$$

On the other hand, we know that

$$\begin{aligned}
D^\gamma Hu(t) \;=\;& \frac{1}{\Gamma(\alpha-\gamma)}\int_0^t(t-s)^{\alpha-\gamma-1}[\sigma g(s,u(s),D^\gamma u(s))-\lambda f(u(s),u''(s))]ds\\
&+\sum_{i=1}^{\infty}\nu_i\frac{1}{\Gamma(\alpha-\gamma)}\int_0^t(t-s)^{\alpha-\gamma-1}\left(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}h_i(\tau,u(\tau))d\tau\right)ds\\
&+\frac{1}{\varphi}\left[\frac{2\Lambda_1t^{2-\gamma}}{\Gamma(3-\gamma)}+\frac{\psi_1t^{1-\gamma}}{\Gamma(2-\gamma)}\right]\left[\kappa_3\int_0^\eta\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}[\sigma g(\tau,u(\tau),D^\gamma u(\tau))-\lambda f(u(\tau),\right.\\
&u''(\tau))]d\tau ds+\sum_{i=1}^{\infty}\kappa_3\nu_i\int_0^\eta\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}\left(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\right)d\tau ds\\
&-\frac{1}{\Gamma(\alpha-2)}\int_0^1(1-s)^{\alpha-3}[\sigma g(s,u(s),D^\gamma u(s))-\lambda f(u(s),u''(s))]ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-2)}\\
&\times\left.\int_0^1(1-s)^{\alpha-3}\left(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\right)ds\right]+\frac{1}{\varphi}\left[\frac{2\Lambda_2t^{2-\gamma}}{\Gamma(3-\gamma)}+\frac{\psi_2t^{1-\gamma}}{\Gamma(2-\gamma)}\right]\\
&\left[\kappa_2\int_0^\theta\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}[\sigma g(\tau,u(\tau),D^\gamma u(\tau))-\lambda f(u(\tau),u''(\tau))]d\tau ds\right.\\
&+\sum_{i=1}^{\infty}\kappa_2\nu_i\int_0^\theta\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}\left(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\right)d\tau ds\\
&-\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}[\sigma g(s,u(s),D^\gamma u(s))-\lambda f(u(s),u''(s))]ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-1)}\\
&\times\left.\int_0^1(1-s)^{\alpha-2}\left(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\right)ds\right]+\frac{1}{\varphi}\left[\frac{2\Lambda_3t^{2-\gamma}}{\Gamma(3-\gamma)}+\frac{\psi_3t^{1-\gamma}}{\Gamma(2-\gamma)}\right]\\
&\left[\kappa_1\int_0^\xi\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}[\sigma g(\tau,u(\tau),D^\gamma u(\tau))-\lambda f(u(\tau),u''(\tau))]d\tau ds\right.\\
&+\sum_{i=1}^{\infty}\kappa_1\nu_i\int_0^\xi\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}\left(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\right)d\tau ds
\end{aligned}$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}[\sigma g(s,u(s),D^\gamma u(s))-\lambda f(u(s),u''(s))]ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha)}$$
$$\times\int_0^1 (1-s)^{\alpha-1}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big].$$

Then, based on this and using the same arguments as before, the following inequality

$$\begin{aligned}
&\|D^\gamma Hu-D^\gamma Hv\|_\infty\\
&\le\left[\frac{M|\delta|+N|\lambda|}{\Gamma(\alpha-\gamma+1)}+\frac{O}{\Gamma(2\alpha-\gamma+1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right]\|u-v\|_X\\
&+\frac{1}{|\varphi|}\left[\frac{2|\Lambda_1|}{\Gamma(3-\gamma)}+\frac{|\psi_1|}{\Gamma(2-\gamma)}\right]\left[\Big(M|\delta|+N|\lambda|\Big)\left(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha-1)}\right)\right.\\
&+\left.\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha-1)}\right)\right]\|u-v\|_X\\
&+\frac{1}{|\varphi|}\left[\frac{2|\Lambda_2|}{\Gamma(3-\gamma)}+\frac{|\psi_2|}{\Gamma(2-\gamma)}\right]\left[\Big(M|\delta|+N|\lambda|\Big)\left(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha)}\right)\right.\\
&+\left.\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha)}\right)\right]\|u-v\|_X\\
&+\frac{1}{|\varphi|}\left[\frac{2|\Lambda_3|}{\Gamma(3-\gamma)}+\frac{|\psi_3|}{\Gamma(2-\gamma)}\right]\left[\Big(M|\delta|+N|\lambda|\Big)\left(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}\right)\right.\\
&+\left.\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha+1)}\right)\right]\|u-v\|_X
\end{aligned}\tag{3.2}$$

is valid.
Also, the second derivative of the operator is given by

$$\begin{aligned}
H''u(t) &= \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}[\sigma g(s,u(s),D^\gamma u(s))-\lambda f(u(s),u''(s))]ds\\
&+\sum_{i=1}^{\infty}\nu_i\frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}h_i(\tau,u(\tau))d\tau\Big)ds\\
&+\Big[\frac{2\Lambda_1}{\varphi}\Big]\Big[\kappa_3\int_0^\eta\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\sigma g(\tau,u(\tau),D^\gamma u(\tau))-\lambda f(u(\tau),u''(\tau))]d\tau ds\\
&+\sum_{i=1}^{\infty}\kappa_3\nu_i\int_0^\eta\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds\\
&-\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}[\sigma g(s,u(s),D^\gamma u(s))-\lambda f(u(s),u''(s))]ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-2)}\\
&\times\int_0^1 (1-s)^{\alpha-3}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big]+\Big[\frac{2\Lambda_2}{\varphi}\Big]\\
&\Big[\kappa_2\int_0^\theta\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\sigma g(\tau,u(\tau),D^\gamma u(\tau))-\lambda f(u(\tau),u''(\tau))]d\tau ds\\
&+\sum_{i=1}^{\infty}\kappa_2\nu_i\int_0^\theta\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds\\
&-\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}[\sigma g(s,u(s),D^\gamma u(s))-\lambda f(u(s),u''(s))]ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-1)}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^1 (1-s)^{\alpha-2}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} H_i(\tau,u(\tau))d\tau\Big)ds\Big] + \Big[\frac{2\Lambda_3}{\varphi}\Big]\\
&\Big[\kappa_1 \int_0^{\xi} \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\sigma g(\tau,u(\tau),D^{\gamma}u(\tau)) - \lambda f(u(\tau),u''(\tau))]d\tau ds\\
&+\sum_{i=1}^{\infty}\kappa_1\nu_i\int_0^{\xi}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds\\
&-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}[\sigma g(s,u(s),D^{\gamma}u(s)) - \lambda f(u(s),u''(s))]ds - \sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha)}\\
&\times \int_0^1 (1-s)^{\alpha-1}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big].
\end{aligned}
$$

Using the above quantity, we obtain

$$
\begin{aligned}
\|H''u - H''v\|_{\infty} \leq\ & \Big[\frac{M|\delta| + N|\lambda|}{\Gamma(\alpha-1)} + \frac{O}{\Gamma(2\alpha-1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_{\infty}\Big]\|u-v\|_X + \frac{2|\Lambda_1|}{|\varphi|}\Big[\Big(M|\delta|\\
& + N|\lambda|\Big)\Big(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha-1)}\Big) + \Big(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_{\infty}\Big)\Big(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)}\\
& + \frac{1}{\Gamma(2\alpha-1)}\Big)\Big]\|u-v\|_X + \frac{2|\Lambda_2|}{|\varphi|}\Big[\Big(M|\delta| + N|\lambda|\Big)\Big(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha)}\Big)\\
& + \Big(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_{\infty}\Big)\Big(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha)}\Big)\Big]\|u-v\|_X\\
& + \frac{2|\Lambda_3|}{|\varphi|}\Big[\Big(M|\delta| + N|\lambda|\Big)\Big(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\Big) + \Big(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_{\infty}\Big)\\
& \times\Big(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha+1)}\Big)\Big]\|u-v\|_X.
\end{aligned}
\tag{3.3}
$$

From (3.1), (3.2) and (3.3) we conclude that

$$
\|Hu - Hv\|_X \leq \Big(\Upsilon_1 + \Upsilon_2 + \Upsilon_3\Big)\|u-v\|_X.
$$

With Banach contraction principle and the condition on Υ, we have H is contractive function, so, H admits a unique fixed point x_0. The proof is thus complete.

4. Examples

In this section, we present two examples to illustrate the validity of the above uniqueness result.

Example 2. *We consider the following problem:*

$$\begin{cases} D^{\frac{5}{2}}u(t)+\frac{1}{2}\frac{|u(t)+u''(t)|}{10\pi(1+|u(t)+u''(t)|)}=\frac{1}{20}\Big(\frac{\sin(u(t))}{e^{t^2+6}}+\frac{|D^{\frac{3}{2}}u(t)|}{200(1+|D^{\frac{3}{2}}u(t)|)}+\big|ln(t)\big|\Big)\\ \qquad +\sum_{i=1}^{\infty}\frac{3e^{-it^2}}{125(i\pi)^2}I^{\frac{5}{2}}\Big(\frac{|u(t)|}{300\big[(t^2+1)+|u(t)|\big]}\Big), t\in(0,1],\\ u(0)+u(1)=\int_0^{0.5}2u(s)ds,\\ u'(0)+u'(1)=\int_0^{0.3}3u(s)ds,\\ u''(0)+u''(1)=\int_0^{0.1}4u(s)ds, \end{cases}$$

where

$$\alpha=\tfrac{5}{2},\ \lambda=\tfrac{1}{2},\ \delta=\tfrac{1}{20},\ \gamma=\tfrac{3}{2},$$

$$\Upsilon_1=0.0416,\ \ \Upsilon_2=0.1550,\ \ \Upsilon_3=0.0397,$$

$$\Upsilon=\Upsilon_1+\Upsilon_2+\Upsilon_3=0.2363.$$

So, thanks to Theorem 1, we confirm that this example has a unique solution.

Example 3. *As a second illustrative example, we consider the problem:*

$$\begin{cases} D^{2.1}u(t)+\frac{3}{10}\frac{|2u(t)+2u''(t)|}{\pi^4(t+2)(1+3|u(t)+u''(t)|)}=\frac{3}{2}\Big(\frac{e^t+\sin(u(t))}{30(t^2+1)}+\frac{|D^{1.2}u(t)|}{20e^{t+1}(1+|D^{1.2}u(t)|)}+\frac{1}{t}\Big)\\ \qquad +\sum_{i=1}^{\infty}\frac{e^{-it^2}}{50i^2}I^{2.1}\Big(\frac{|u(t)|}{200\big[(t+1)+|u(t)|\big]}+e^t\Big), t\in(0,1],\\ u(0)+u(1)=\int_0^{0.1}u(s)ds,\\ u'(0)+u'(1)=\int_0^{0.5}2u(s)ds,\\ u''(0)+u''(1)=\int_0^{0.3}u(s)ds, \end{cases}$$

where

$$\alpha=2.1,\ \lambda=\tfrac{3}{10},\ \delta=\tfrac{3}{2},\ \gamma=1.2,$$

$$\Upsilon_1=0.1498,\ \ \Upsilon_2=0.4170,\ \ \Upsilon_3=0.0911,$$

$$\Upsilon=\Upsilon_1+\Upsilon_2+\Upsilon_3=0.6579.$$

Also, by Theorem 1, our example has a unique solution.

5. Stability for Unique Solutions

Definition 3. *The equation (1.1) has the Ulam Hyers stability if there exists a real number $\Theta>0$, such that for each $\varepsilon>0, t\in]0,1]$ and for each $u\in X$ solution of the inequality*

$$\Big|D^{\alpha}u(t)+\lambda f(u(t),u''(t))-\sigma g(t,u(t),D^{\gamma}u(t))-\sum_{i=1}^{\infty}\nu_i\Phi_i(t)I^{\alpha}h_i(t,u(t))\Big|\le\varepsilon, \tag{5.1}$$

there exists $v\in X$ a solution of (1.1), such that

$$\|u-v\|_X\le\Theta\varepsilon.$$

Definition 4. *The equation* (1.1) *has the Ulam Hyers stability in the generalized sense if there exists* $\Omega \in C(\mathbb{R}^+, \mathbb{R}^+)$; $\Omega(0) = 0$, *such that for each* $\varepsilon > 0$, *and for any* $u \in X$ *solution of* (5.1), *there exists a solution* $v \in X$ *of (1.1), such that*

$$\|u - v\|_X < \Omega(\varepsilon).$$

Now, we are able to prove the first main result.

Theorem 4. *Under the conditions of* (1), *problem* (1.1) *is Ulam Hyers stable.*

Proof: Let $u \in X$ be a solution of (5.1), and let, by Theorem 1, $v \in X$ be the unique solution of (1.1). By integration of (5.1), we obtain

$$\begin{aligned}
&\Big|u(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\sigma g(s,u(s),D^\gamma u(s)) - \lambda f(u(s),u''(s))]ds \\
&-\sum_{i=1}^{\infty} \nu_i \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} h_i(\tau,u(\tau))d\tau\Big)ds \\
&-\Big[\frac{\Lambda_1 t^2 + \psi_1 t + \Delta_1}{\varphi}\Big]\Big[\kappa_3 \int_0^\eta \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\sigma g(\tau,u(\tau),D^\gamma u(\tau)) - \lambda f(u(\tau),u''(\tau))]d\tau ds \\
&+\sum_{i=1}^{\infty}\kappa_3 \nu_i \int_0^\eta \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}[\sigma g(s,u(s),D^\gamma u(s)) - \lambda f(u(s),u''(s))]ds - \sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-2)} \\
&\times \int_0^1 (1-s)^{\alpha-3}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big] - \Big[\frac{\Lambda_2 t^2 + \psi_2 t + \Delta_2}{\varphi}\Big] \\
&\Big[\kappa_2 \int_0^\theta \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\sigma g(\tau,u(\tau),D^\gamma u(\tau)) - \lambda f(u(\tau),u''(\tau))]d\tau ds \\
&+\sum_{i=1}^{\infty}\kappa_2 \nu_i \int_0^\theta \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}[\sigma g(s,u(s),D^\gamma u(s)) - \lambda f(u(s),u''(s))]ds - \sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-1)} \\
&\times \int_0^1 (1-s)^{\alpha-2}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big] - \Big[\frac{\Lambda_3 t^2 + \psi_3 t + \Delta_3}{\varphi}\Big] \\
&\Big[\kappa_1 \int_0^\xi \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\sigma g(\tau,u(\tau),D^\gamma u(\tau)) - \lambda f(u(\tau),u''(\tau))]d\tau ds \\
&+\sum_{i=1}^{\infty}\kappa_1 \nu_i \int_0^\xi \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}[\sigma g(s,u(s),D^\gamma u(s)) - \lambda f(u(s),u''(s))]ds - \sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha)} \\
&\times \int_0^1 (1-s)^{\alpha-1}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big]\Big| \le I^\alpha \varepsilon(t).
\end{aligned} \tag{5.2}$$

Using (5.1) and (5.2), we get

$$
\begin{aligned}
\|u-v\|_\infty \leq{}& I^\alpha \varepsilon(t)+\left[\frac{M|\delta|+N|\lambda|}{\Gamma(\alpha+1)}+\frac{O}{\Gamma(2\alpha+1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right]\|u-v\|_X \\
&+\left[\frac{|\Lambda_1|+|\psi_1|+|\Delta_1|}{|\varphi|}\right]\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha-1)}\right)\right. \\
&\left.+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha-1)}\right)\right]\|u-v\|_X \\
&+\left[\frac{|\Lambda_2|+|\psi_2|+|\Delta_2|}{|\varphi|}\right]\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha)}\right)\right. \\
&\left.+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha)}\right)\right]\|u-v\|_X \\
&+\left[\frac{|\Lambda_3|+|\psi_3|+|\Delta_3|}{|\varphi|}\right]\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}\right)\right. \\
&\left.+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i(t)\|_\infty\right)\left(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha+1)}\right)\right]\|u-v\|_X.
\end{aligned}
\tag{5.3}
$$

Therefore, we have

$$\|u-v\|_\infty \quad \leq \quad \frac{\varepsilon}{\Gamma(\alpha+1)(1-\Upsilon_1)} \leq \varepsilon\ \Xi_1.$$

On the other hand, we have

$$\|D^\gamma(u-v)\|_\infty \quad \leq \quad \frac{\varepsilon}{\Gamma(\alpha+1)(1-\Upsilon_2)} \leq \varepsilon\ \Xi_2.$$

Consequently,

$$\|u''-v''\|_\infty \quad \leq \quad \frac{\varepsilon}{\Gamma(\alpha+1)(1-\Upsilon_3)} \leq \varepsilon\ \Xi_3.$$

Thus,

$$\|u-v\|_X \quad \leq \quad \varepsilon\Big(\ \Xi_1+\Xi_2+\Xi_3\Big).$$

Thus, (1.1) has the Ulam Hyers stability.

Remark 1. *In the case* $\Omega(\varepsilon)=\varepsilon\Big(\ \Xi_1+\Xi_2+\Xi_3\Big)$, *we obtain the generalised Ulam Hyers stability for* (1.1).

Remark 2. *The problems of the above two examples are Ulam Hyers stable since they fulfill the conditions of Theorem* 1.
In particular, in both cases, we have proved that there is a solution v, *such that for each* $\varepsilon>0, t\in]0,1]$ *and for each* $u\in X$ *solution of inequality* (5.1), *we can write, for the first example:*

$$\|u-v\|_\infty \leq 0.7849\varepsilon,\ \ \|D^{\frac{3}{2}}(u-v)\|_\infty \leq 0.8902\varepsilon,\ \ \|u''-v''\|_\infty \leq 0.7834\varepsilon.$$

Thus,

$$\|u-v\|_X \quad \leq \quad 2.4585\varepsilon.$$

However, for the second example, we can write

$$\|u-v\|_\infty \leq 1.0675\varepsilon,\ \ \|D^{1.2}(u-v)\|_\infty \leq 1.5568\varepsilon,\ \ \|u''-v''\|_\infty \leq 0.9986\varepsilon.$$

Thus,

$$\|u-v\|_X \quad \leq \quad 3.6229\varepsilon.$$

6. Numerical Simulations

In this paragraph, we illustrate an effective numerical approach to Riemann-Liouville integral and Caputo derivative. We need to recall the approximation theorems of the papers [9, 15]. Based on Caputo derivative approximation, we investigate, for some given parameters, the behavior of the considered problem by studding one of the two proposed examples . In order to do this, we should initially obtain a reduced fractional differential system which can be equivalent to the considered problem. The numerical simulations are then performed using a Runge-Kutta integrator of order 4.

Theorem 5. *Let $y \in \mathcal{C}^1([0,1],\mathbb{R})$. The fractional integration approach is:*

$$J^{\alpha} y(t_i) \simeq \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{i} y(t_j)\, \sigma_j(\alpha), \qquad i = 0, \ldots, n+1,$$

where

$$\sigma_j(\alpha) = \begin{cases} (n+2-j)^{(\alpha+1)} + (n-j)^{(\alpha+1)} - 2(n-j+1)^{(\alpha+1)},\ j = 1 \ldots i-1. \\ (n)^{(\alpha+1)} - (n-\alpha)(n+1)^{\alpha},\ j = 0, and\ 1,\ j = i. \end{cases}$$

Theorem 6. *Let $y \in \mathcal{C}^1([0,1],\mathbb{R})$ and $0 < \alpha \leq 1$. Then, we get:*

$$D^{\alpha} y(t_i) \simeq \frac{h^{1-\alpha}}{\Gamma(1-\alpha+2)} \sum_{j=0}^{i} y^{(j)}(t_j) \sigma_j(1-\alpha), \qquad i = 0, \ldots, n,$$

where,

$$y^{(j)} = \begin{cases} \dfrac{y_1 - y_0}{h},\ j = 0, \quad \dfrac{y_{j+1} - y_{j-1}}{2h},\ j = 1 \ldots i-1, \quad \dfrac{y_i - y_{i-1}}{h},\ j = i. \end{cases}$$

Remark 3. *The problem* (1.1) *can be reduced to the formula below:*

$$\begin{aligned} &D^1 u(t) = v(t) \\ &D^1 v(t) = w(t) \\ &D^1 w(t) = D^{3-\alpha}\Big(-\lambda f(u(t), u''(t)) + \delta g(t, u(t), D^{\gamma} u(t)) + \sum_{i=1}^{\infty} \nu_i \Phi_i(t) I^{\alpha} h_i(t, u(t)) \Big). \end{aligned}$$

Through numerical simulations achieved by a combination of Caputo approach and the fourth-order Runge-Kutta method on the first example, we obtain:

Figure 1. Solution for the first example, on the plan u-w, for four values of α.

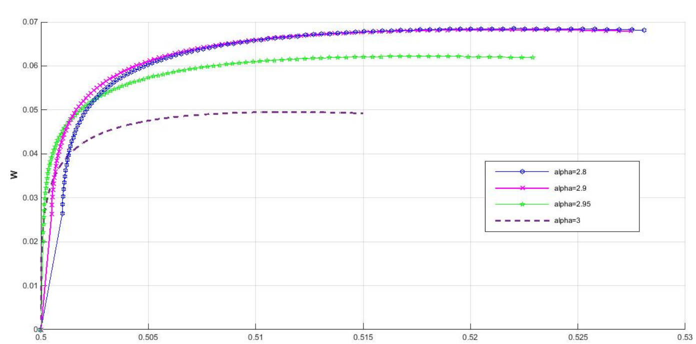


Figure 2. Behavior of the dynamics of the first example, on the plan v-w, for four values of α.

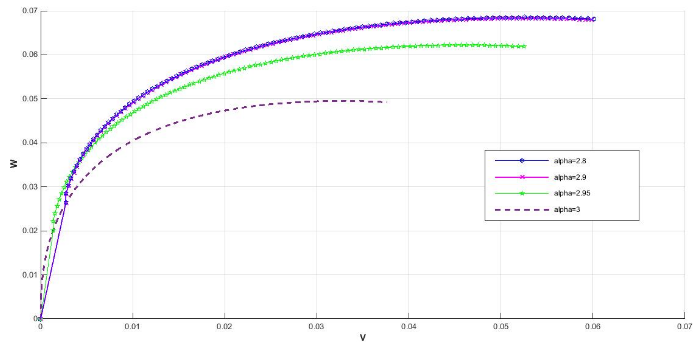


Figure 3. Behavior of the solution for the first example, for different values of α.

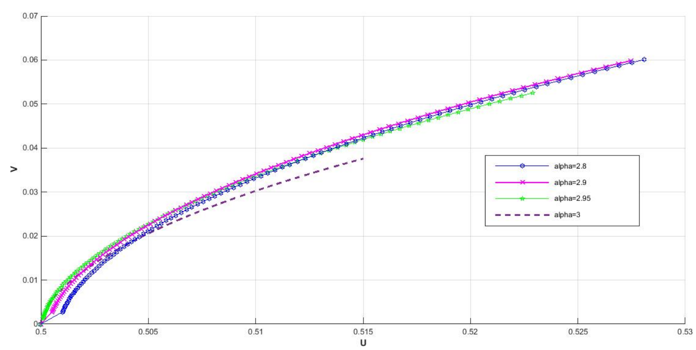
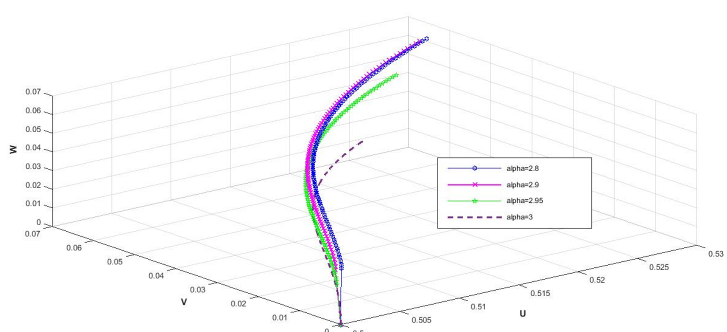


Figure 4. 3D representation for the solution of the first example, for different values of α.



7. Conclusion

A new type of nonlinear integro-differential equations involving convergent series for Riemann Liouville integrals and some other functions has been investigated. The considered problem has a time variable singularity. By application of Banach fixed point theorem, we have established an existence and uniqueness result, then, we have discussed the Ulam-Hyers stability for the problem. Two illustrative examples have been discussed. Another interesting point that has been discussed in this work is the application of a Caputo derivative approximation; by using the Rung Kutta method, the approximation has allowed us to present a numerical study with some time and space graphs for one of our examples.

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